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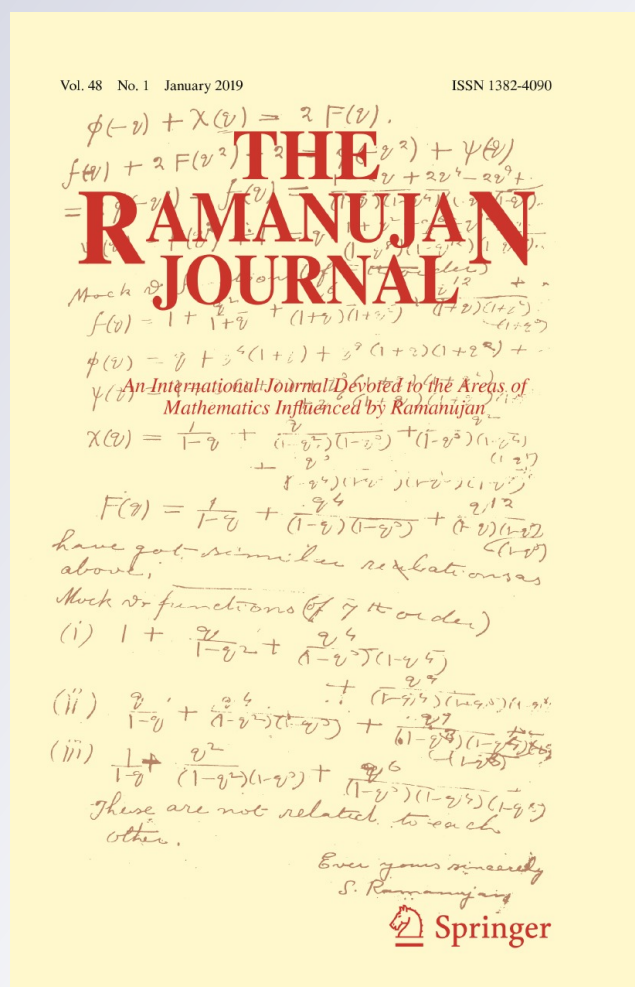
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Generalization of Taylor's formula and differential transform method for composite fractional q -derivative

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Abstract In the present paper, we first establish a generalized q -Taylor's formula involving composite fractional q -derivative. Next, we define the generalized q -differential transform and its inverse for composite fractional q -derivative and establish some basic properties for this transform. We also illustrate the effectiveness of these results by solving two fractional q -difference equations.

Keywords q -Taylor's formula · Composite fractional q -Derivative · q -Differential transform method · q -Difference equation

Mathematics Subject Classification 26A33 · 33E12 · 34A12 · 35A22 · 33D90

1 Introduction

Differential equations of fractional order appear more and more frequently in various research areas and engineering applications. An effective and easy to use method

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for solving such equations is needed. One such method developed by Odibat et al. [8] is the generalized differential transform method for fractional differential equations involving Caputo fractional derivative, which is a generalization of differential transform method introduced by [11].

The fractional q -difference calculus had its origin in the works of [2] and [1]. The concept of the q -differential transform method was first proposed by [7] for solving ordinary q -difference equations. This method is further extended to solve fractional q -difference equations involving Caputo fractional q -derivative by Garg et al. [4]. In this paper, we develop a new generalization of q -differential transform which is based on generalized q -Taylor's formula and Composite fractional q -derivative.

The paper is organized as follows. In Sect. 2, we provide definitions which will be used in subsequent sections. In Sect. 3, we first define generalized composite fractional q -derivative $D_{q,a}^{\alpha,\beta}$. Then, we obtain some basic results for the composite fractional q -derivative $D_q^{\alpha,\beta}$ considered in the spaces defined in Sect. 2. In Sect. 4, we establish generalized q -Taylor's formula involving composite fractional q -derivative. In Sect. 5, we define generalized q -differential transform and its inverse and establish some basic properties for this transform. In Sect. 6, we solve two fractional q -difference equations involving composite fractional q -derivative.

2 Preliminaries and definitions

For the completeness, we mention the following definitions [3].

Definition 2.1 For $0 < |q| < 1$, a subset A of $\mathbb{C}$ is called q -geometric if $qz \in A$ whenever $z \in A$. If a subset A of $\mathbb{C}$ is q -geometric, then it contains all geometric sequences $\{zq^n\}_{n=0}^{\infty}$, $z \in A$.

Definition 2.2 A function f which is defined on a q -geometric set A , $0 \in A$, is said to be q -regular at zero if $\lim_{n \rightarrow \infty} f(zq^n) = f(0)$ for all $z \in A$.

Definition 2.3 By $L_q^1(0, a)$, $a > 0$, we mean the Banach space of all the functions defined on $(0, a]$ such that

$$\|f\| = \int_0^a |f(t)| d_q t < \infty. \quad (2.1)$$

Two functions in $L_q^1(0, a)$, $a > 0$ are considered to be the same function if they have the same values at the sequence $\{aq^n\}_{n=0}^{\infty}$.

Let $L_q^1[0, a]$ denote the space of all functions f defined on $(0, a]$ such that $f \in L_q^1(0, x)$ for all $x \in (qa, a]$.

Definition 2.4 By $\mathcal{AC}_q[0, a]$ we mean the space of all functions f defined on $[0, a]$ such that f is q -regular at zero and

$$\sum_{j=0}^{\infty} \left| f(tq^j) - f(tq^{j+1}) \right| < \infty, \quad t \in (qa, a]. \quad (2.2)$$

The space $\mathcal{AC}_q^{(m)}[0, a]$, $m \in \mathbb{N}$ is the space of all the functions f defined on $[0, a]$ such that $f, D_q f, \dots, D_q^{m-1} f$ are q -regular at zero and $D_q^{m-1} f \in \mathcal{AC}_q[0, a]$.

The Riemann–Liouville fractional q -integral is defined by Al-Salam in [2] as

$$I_q^\alpha f(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x; q)_{\alpha-1} f(t) d_q t; \quad \alpha \notin \{-1, -2, \dots\}, \quad x > 0. \quad (2.3)$$

Further in [10], the generalized Riemann–Liouville fractional q -integral operator for $\alpha \in \mathbb{R}^+$ is given as

$$I_{q,a}^\alpha f(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_a^x (qt/x; q)_{\alpha-1} f(t) d_q t; \quad a < x. \quad (2.4)$$

We shall require the following semi-group property

$$I_q^\beta I_q^\alpha f(x) = I_q^{\alpha+\beta} f(x), \quad x > 0, \quad \alpha, \beta \in \mathbb{R}^+, \quad (2.5)$$

and the image of power function

$$I_q^\alpha x^{\lambda-1} = \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\alpha)} x^{\lambda+\alpha-1}, \quad x > 0, \quad \lambda > -1, \quad \alpha \in \mathbb{R}^+. \quad (2.6)$$

The Riemann–Liouville type fractional q -derivative of order α , $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, is defined as [10]

$$D_{q,a}^\alpha f(x) = D_{q,a}^m I_{q,a}^{m-\alpha} f(x); \quad \alpha \in \mathbb{R}^+, \quad a < x, \quad (2.7)$$

and $D_{q,0}^\alpha f(x) = D_q^\alpha f(x)$.

The Caputo type fractional q -derivative, for $m-1 < \alpha \leq m$, $m \in \mathbb{N}$ is defined as [9].

$$*_D_{q,a}^\alpha f(x) = I_{q,a}^{m-\alpha} D_{q,a}^m f(x); \quad \alpha \in \mathbb{R}^+, \quad a < x, \quad (2.8)$$

and $*D_{q,0}^\alpha f(x) = *_D_q^\alpha f(x)$.

Following [3], we say that $D_q^\alpha f(x)$ exists, if $f \in L_q^1[0, a]$ and $I_q^{m-\alpha} f \in \mathcal{AC}_q^{(m)}[0, a]$ and $*D_q^\alpha f(x)$ exists if $f \in \mathcal{AC}_q^{(m)}[0, a]$.

We shall also require the following composition formula [10]

$$I_q^\alpha D_q^\alpha f(x) = f(x) - \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \left[I_q^{1-\alpha} f(0) \right], \quad 0 < \alpha \leq 1, \quad x > 0. \quad (2.9)$$

Mean value theorem for q -integrals [5]

Let f be a continuous function defined on $[0, a]$, and g be a nonnegative function defined on $[0, a]$ and q -regular at zero. Then $\exists c \in (0, a)$ such that

$$\int_0^a f(t)g(t)d_q t = f(c) \int_0^a g(t)d_q t. \quad (2.10)$$

3 Generalized composite fractional q -derivative

We define the generalized composite fractional q -derivative of order α , $0 < \alpha \leq 1$ and type β , $0 \leq \beta \leq 1$, as follows

$$D_{q,a}^{\alpha,\beta} f(x) = I_{q,a}^{\beta(1-\alpha)} D_q I_{q,a}^{(1-\beta)(1-\alpha)} f(x), \quad a < x, \quad 0 < |q| < 1. \quad (3.1)$$

For $\beta = 0$, it reduces to the Riemann–Liouville type fractional q -derivative of order α defined by (2.7), and for $\beta = 1$, it reduces to the Caputo type fractional q -derivative of order α defined by (2.8).

For $0 < \beta < 1$, it interpolates continuously between the Riemann–Liouville type fractional q -derivative of order α and the Caputo type fractional q -derivative of order α .

This $D_{q,a}^{\alpha,\beta}$ is a q -extension of the generalized Riemann–Liouville fractional derivative $D_x^{\alpha,\beta}$ defined by Hilfer [6].

In the subsequent work, we shall study the following composite fractional q -derivative of order α , $0 < \alpha \leq 1$ and type β , $0 \leq \beta \leq 1$

$$D_q^{\alpha,\beta} f(x) = I_q^{\beta(1-\alpha)} D_q I_q^{(1-\beta)(1-\alpha)} f(x), \quad x > 0, \quad 0 < |q| < 1. \quad (3.2)$$

Following the existence conditions for $D_q^\alpha f(x)$ and $*D_q^\alpha f(x)$ as given in [3], we say that $D_q^{\alpha,\beta} f(x)$ exists if $f \in L_q^1[0, a]$, $I_q^{(1-\beta)(1-\alpha)} f \in \mathcal{AC}_q[0, a]$.

We now obtain some results for the composite fractional q -derivative $D_q^{\alpha,\beta}$.

Theorem 3.1 For $f \in L_q^1[0, a]$, $I_q^{(1-\beta)(1-\alpha)} f \in \mathcal{AC}_q[0, a]$, $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 < |q| < 1$, $x > 0$ and $\lambda > 0$, we have the image of power function under $D_q^{\alpha,\beta}$ as follows

$$D_q^{\alpha,\beta} x^{\lambda-1} = \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda - \alpha)} x^{\lambda-\alpha-1}. \quad (3.3)$$

Proof In view of definition (3.2) and the result (2.6), we get

$$\begin{aligned} D_q^{\alpha,\beta} x^{\lambda-1} &= I_q^{\beta(1-\alpha)} D_q \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda + (1-\beta)(1-\alpha))} x^{\lambda+(1-\beta)(1-\alpha)-1} \\ &= I_q^{\beta(1-\alpha)} \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda + (1-\beta)(1-\alpha) - 1)} x^{\lambda+(1-\beta)(1-\alpha)-2}. \end{aligned}$$

Using result (2.6) again, we get (3.3). $\square$

Theorem 3.2 For $f \in L_q^1[0, a]$, $I_q^{(1-\beta)(1-\alpha)} f \in \mathcal{AC}_q[0, a]$, $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 < |q| < 1$, the composition of Riemann–Liouville type fractional q -integral (2.4) with the composite fractional q -derivative (3.2) is given by

$$I_q^\alpha D_q^{\alpha, \beta} f(x) = f(x) - \frac{x^{-(1-\alpha)(1-\beta)}}{\Gamma_q(1 - (1-\alpha)(1-\beta))} I_q^{(1-\alpha)(1-\beta)} f(0). \quad (3.4)$$

Proof From (3.2), we can write

$$D_q^{\alpha, \beta} f(t) = I_q^{\beta(1-\alpha)} D_q I_q^{(1-\beta)(1-\alpha)} f(t) = I_q^{\beta(1-\alpha)} D_q^{\alpha+\beta(1-\alpha)} f(t).$$

Applying I_q^α on both the sides and using the semi-group property (2.5), we get

$$I_q^\alpha D_q^{\alpha, \beta} f(t) = I_q^{\alpha+\beta(1-\alpha)} D_q^{\alpha+\beta(1-\alpha)} f(t). \quad (3.5)$$

In view of the result (2.9), we arrive at (3.4). $\square$

4 Generalized q -Taylor's formula

Our main result of this section is generalized q -Taylor's formula involving $D_q^{\alpha, \beta}$, given as Theorem 4.1. To prove it, we first establish two lemmas as follows.

Lemma 1 (Generalized mean value theorem) If $f \in L_q^1[0, a]$, $I_q^{(1-\beta)(1-\alpha)} f \in \mathcal{AC}_q[0, a]$, $D_q^{\alpha, \beta} f \in C[0, a]$, $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 < |q| < 1$ and $x \in (0, a]$, then $\exists c \in (0, x)$, such that

$$f(x) = \frac{x^{-(1-\alpha)(1-\beta)}}{\Gamma_q(1 - (1-\alpha)(1-\beta))} I_q^{(1-\alpha)(1-\beta)} f(0) + \frac{1}{\Gamma_q(\alpha + 1)} D_q^{\alpha, \beta} f(c) x^\alpha. \quad (4.1)$$

Proof On applying Riemann–Liouville fractional q -integral I_q^α given by (2.4), to $D_q^{\alpha, \beta} f(x)$, we get

$$I_q^\alpha D_q^{\alpha, \beta} f(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x; q)_{\alpha-1} D_q^{\alpha, \beta} f(t) d_q t. \quad (4.2)$$

Using second mean value theorem for q -integral given by (2.10), for $c \in (0, x)$, we get

$$I_q^\alpha D_q^{\alpha, \beta} f(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} D_q^{\alpha, \beta} f(c) \int_0^x (qt/x; q)_{\alpha-1} d_q t. \quad (4.3)$$

Using (2.4) and (2.6), with $\lambda = 1$, it is simplified to

$$I_q^\alpha D_q^{\alpha, \beta} f(x) = \frac{1}{\Gamma_q(\alpha + 1)} D_q^{\alpha, \beta} f(c) x^\alpha. \quad (4.4)$$

Using (3.4), we get (4.1). $\square$

Lemma 2 If $D_q^{k\alpha, \beta} f \in L_q^1[0, a]$ and $I_q^{(1-\alpha)(1-\beta)} D_{q,a}^{k\alpha, \beta} f \in \mathcal{AC}_q[0, a]$, $k = 0, 1, \dots, n+1$, $n \in \mathbb{N}$, then for $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 < |q| < 1$, we have

$$\begin{aligned} I_q^{n\alpha} D_q^{n\alpha, \beta} f(x) - I_q^{(n+1)\alpha} D_q^{(n+1)\alpha, \beta} f(x) \\ = \frac{x^{n\alpha - (1-\alpha)(1-\beta)}}{\Gamma_q(n\alpha - (1-\alpha)(1-\beta) + 1)} I_q^{(1-\alpha)(1-\beta)} D_q^{n\alpha, \beta} f(0), \end{aligned} \quad (4.5)$$

where $D_q^{n\alpha, \beta} = D_q^{\alpha, \beta} \cdot D_q^{\alpha, \beta} \cdots D_q^{\alpha, \beta}$ (n times).

Proof The left side of (4.5), using (2.5), can be written as

$$I_q^{n\alpha} \left[D_q^{n\alpha, \beta} f(x) - I_q^\alpha D_q^{(n+1)\alpha, \beta} f(x) \right] = I_q^{n\alpha} \left[D_q^{n\alpha, \beta} f(x) - I_q^\alpha D_q^{\alpha, \beta} D_q^{n\alpha, \beta} f(x) \right]. \quad (4.6)$$

Using (3.4), it becomes

$$\begin{aligned} &= I_q^{n\alpha} \left[D_q^{n\alpha, \beta} f(x) - D_q^{n\alpha, \beta} f(x) + \frac{x^{-(1-\alpha)(1-\beta)}}{\Gamma_q(1 - (1-\alpha)(1-\beta))} I_q^{(1-\alpha)(1-\beta)} D_q^{n\alpha, \beta} f(0) \right] \\ &= I_q^{n\alpha} \left[\frac{x^{-(1-\alpha)(1-\beta)}}{\Gamma_q(1 - (1-\alpha)(1-\beta))} I_q^{(1-\alpha)(1-\beta)} D_q^{n\alpha, \beta} f(0) \right]. \end{aligned}$$

Finally, using result (2.6), we get right side of (4.5). $\square$

Theorem 4.1 If $D_q^{k\alpha, \beta} f \in L_q^1[0, a]$, $I_q^{(1-\alpha)(1-\beta)} D_{q,a}^{k\alpha, \beta} f \in \mathcal{AC}_q[0, a]$, $k = 0, 1, \dots, n+1$, $n \in \mathbb{N}$, $D_q^{(n+1)\alpha, \beta} f \in C[0, a]$ and $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 < |q| < 1$, then $\exists c \in (0, x)$ such that

$$\begin{aligned} f(x) &= \sum_{k=0}^n \frac{x^{k\alpha - (1-\alpha)(1-\beta)}}{\Gamma_q(k\alpha - (1-\alpha)(1-\beta) + 1)} I_q^{(1-\alpha)(1-\beta)} D_q^{k\alpha, \beta} f(0) \\ &\quad + \frac{D_q^{(n+1)\alpha, \beta} f(c)}{\Gamma_q((n+1)\alpha + 1)} x^{(n+1)\alpha}, \end{aligned} \quad (4.7)$$

where $D_q^{n\alpha, \beta} = D_q^{\alpha, \beta} D_q^{\alpha, \beta} \cdots D_q^{\alpha, \beta}$ (n times).

Proof From (4.5), we have

$$\begin{aligned} &\sum_{k=0}^n \left\{ \left(I_q^{k\alpha} D_q^{k\alpha, \beta} f \right) (x) - \left(I_q^{(k+1)\alpha} D_q^{(k+1)\alpha, \beta} f \right) (x) \right\} \\ &= \sum_{k=0}^n \frac{x^{k\alpha - (1-\alpha)(1-\beta)}}{\Gamma_q(k\alpha - (1-\alpha)(1-\beta) + 1)} I_q^{(1-\alpha)(1-\beta)} D_q^{k\alpha, \beta} f(0). \end{aligned} \quad (4.8)$$

On simplification, it becomes $\square$

$$f(x) = \sum_{k=0}^n \frac{x^{k\alpha - (1-\alpha)(1-\beta)}}{\Gamma_q(k\alpha - (1-\alpha)(1-\beta) + 1)} I_q^{(1-\alpha)(1-\beta)} D_q^{k\alpha, \beta} f(0) + \left(I_q^{(n+1)\alpha} D_q^{(n+1)\alpha, \beta} f \right)(x). \quad (4.9)$$

Using definition (2.4), we get

$$f(x) = \sum_{k=0}^n \frac{x^{k\alpha - (1-\alpha)(1-\beta)}}{\Gamma_q(k\alpha - (1-\alpha)(1-\beta) + 1)} I_q^{(1-\alpha)(1-\beta)} D_q^{k\alpha, \beta} f(0) + \frac{x^{(n+1)\alpha - 1}}{\Gamma_q((n+1)\alpha)} \int_0^x (qt/x; q)_{(n+1)\alpha - 1} D_q^{(n+1)\alpha, \beta} f(t) d_q t. \quad (4.10)$$

Using Mean Value Theorem for q -integral given by (2.10), then applying (2.6) (with $\lambda = 1$), we get

$$f(x) = \sum_{k=0}^n \frac{x^{k\alpha - (1-\alpha)(1-\beta)}}{\Gamma_q(k\alpha - (1-\alpha)(1-\beta) + 1)} I_q^{(1-\alpha)(1-\beta)} D_q^{k\alpha, \beta} f(0) + \frac{D_q^{(n+1)\alpha, \beta} f(c)}{\Gamma_q((n+1)\alpha + 1)} x^{(n+1)\alpha}; \quad c \in (0, x). \quad (4.11)$$

If the remainder term $\frac{D_q^{(n+1)\alpha, \beta} f(c)}{\Gamma_q((n+1)\alpha + 1)} x^{(n+1)\alpha} \rightarrow 0$ as $n \rightarrow \infty$, we can write generalized q -Taylor's formula involving composite fractional q -derivatives as

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{k\alpha - (1-\alpha)(1-\beta)}}{\Gamma_q(k\alpha - (1-\alpha)(1-\beta) + 1)} I_q^{(1-\alpha)(1-\beta)} D_q^{k\alpha, \beta} f(0). \quad (4.12)$$

Remark If we take $\beta = 0$ and $\beta = 1$ in Lemmas 1, 2 and Theorem 4.1, we arrive at the results essentially similar to the results proved in [5] as Theorem 3.3, Lemma 4.1 and Theorem 4.2, respectively, for Riemann–Liouville type fractional q -derivative and Caputo type fractional q -derivative.

5 Generalized q -differential transform

On making use of generalized q -Taylor's formula obtained in Sect. 4, for $D_q^{k\alpha, \beta} \in L_q^1[0, a]$, $I_q^{(1-\alpha)(1-\beta)} D_q^{k\alpha, \beta} f \in \mathcal{AC}_q[0, a]$, $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 < |q| < 1$, we define the generalized q -differential transform $F_{q, \alpha, \beta}(k)$ of function $f(x)$ at point $x = 0$ as follows:

$$F_{q, \alpha, \beta}(k) = \frac{1}{\Gamma_q(\alpha k - (1-\alpha)(1-\beta) + 1)} \left[I_q^{(1-\alpha)(1-\beta)} D_q^{k\alpha, \beta} f(0) \right], \quad (5.1)$$

where $D_q^{k\alpha,\beta} = D_q^{\alpha,\beta} D_q^{\alpha,\beta} \cdots D_q^{\alpha,\beta}$ (k times).

The inverse generalized q -differential transform of $F_{q,\alpha,\beta}(k)$ in view of (4.12) is given by

$$f(x) = \sum_{k=0}^{\infty} F_{q,\alpha,\beta}(k) x^{k\alpha - (1-\alpha)(1-\beta)}. \quad (5.2)$$

Some basic properties of the generalized q -differential transform are as given below:

Theorem 5.1 *If $F_{q,\alpha,\beta}(k)$, $U_{q,\alpha,\beta}(k)$ and $V_{q,\alpha,\beta}(k)$ are generalized q -differential transforms of functions $f(x)$, $u(x)$ and $v(x)$ at point $x = 0$, respectively, then the following results hold.*

- (a) *If $f(x) = u(x) \pm v(x)$, then $F_{q,\alpha,\beta}(k) = U_{q,\alpha,\beta}(k) \pm V_{q,\alpha,\beta}(k)$.*
- (b) *If $f(x) = cu(x)$, c is constant, then $F_{q,\alpha,\beta}(k) = cU_{q,\alpha,\beta}(k)$.*
- (c) *If $f(x) = x^{n\alpha - (1-\alpha)(1-\beta)}$, $n \in \mathbb{N}$, then $F_{q,\alpha,\beta}(k) = \delta(k - n)$, where*

$$\delta(k) = \begin{cases} 1, & \text{when } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

- (d) *If $f(x) = D_q^{\alpha,\beta} u(x)$, $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, then*

$$F_{q,\alpha,\beta}(k) = \frac{\Gamma_q(\alpha(k+1) - (1-\alpha)(1-\beta) + 1)}{\Gamma_q(\alpha k - (1-\alpha)(1-\beta) + 1)} U_{q,\alpha,\beta}(k+1).$$

- (e) *If $f(x) = u(cx)$, c is constant, then*

$$F_{q,\alpha,\beta}(k) = c^{k\alpha - (1-\alpha)(1-\beta)} U_{q,\alpha,\beta}(k). \quad (5.3)$$

Proof By linearity of generalized q -differential transform, we can easily obtain results (a) and (b).

(c) Writing

$$f(x) = \sum_{k=0}^{\infty} \delta(k - n) x^{k\alpha - (1-\alpha)(1-\beta)}. \quad (5.4)$$

We get in view of (5.2), $F_{q,\alpha,\beta}(k) = \delta(k - n)$.

(d) Taking $f(x) = D_{q,a}^{\alpha,\beta} u(x)$ in (5.1), we get

$$\begin{aligned} F_{q,\alpha,\beta}(k) &= \frac{1}{\Gamma_q(\alpha k - (1-\alpha)(1-\beta) + 1)} \left[I_q^{(1-\alpha)(1-\beta)} D_q^{k\alpha,\beta} \left(D_q^{\alpha,\beta} u(x) \right) \right]_{x=0}, \\ &= \frac{1}{\Gamma_q(\alpha k - (1-\alpha)(1-\beta) + 1)} \left[I_q^{(1-\alpha)(1-\beta)} D_q^{(k+1)\alpha,\beta} u(x) \right]_{x=0}, \\ &= \frac{\Gamma_q(\alpha(k+1) - (1-\alpha)(1-\beta) + 1)}{\Gamma_q(\alpha k - (1-\alpha)(1-\beta) + 1)} U_{q,\alpha,\beta}(k+1). \end{aligned}$$

(e) On replacing x by cx in inverse generalized q -differential transform (5.2) for $u(x)$, we get

$$u(cx) = \sum_{k=0}^{\infty} U_{q,\alpha,\beta}(k)(cx)^{k\alpha-(1-\alpha)(1-\beta)} = \sum_{k=0}^{\infty} U_{q,\alpha,\beta}(k)(cx)^{k\alpha-(1-\alpha)(1-\beta)}, \quad (5.5)$$

which again in view of (5.2) gives the result(5.3). $\square$

6 Applications

In this section, we shall apply the generalized q -differential transform, to solve two fractional q -difference equations involving composite fractional q -derivative $D_q^{\alpha,\beta}$.

Problem 1 For $y \in L_q^1[0, a]$, $I_q^{(1-\beta)(1-\alpha)}y \in \mathcal{AC}_q[0, a]$, $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 < |q| < 1$, let us consider the following q -initial value problem

$$D_q^{\alpha,\beta}y(x) - \lambda y(x) = 0, \quad (6.1)$$

where λ is a real number, with initial condition

$$I_q^{(1-\beta)(1-\alpha)}y(x) \Big|_{x \rightarrow 0^+} = y_0. \quad (6.2)$$

The solution of the above problem is given by

$$y(x) = y_0 x^{-(1-\alpha)(1-\beta)} e_{\alpha,\alpha+\beta(1-\alpha)}(\lambda x^\alpha; q). \quad (6.3)$$

Proof Taking generalized q -differential transform (5.1), of (6.1) and initial condition (6.2), making use of properties of generalized q -differential transform given in Theorem 5.1, we get

$$\frac{\Gamma_q(\alpha(k+1) - (1-\alpha)(1-\beta) + 1)}{\Gamma_q(\alpha k - (1-\alpha)(1-\beta) + 1)} Y_{q,\alpha,\beta}(k+1) - \lambda Y_{q,\alpha,\beta}(k) = 0 \quad (6.4)$$

and

$$Y_{q,\alpha,\beta}(0) = \frac{y_0}{\Gamma_q(1 - (1-\alpha)(1-\beta))}. \quad (6.5)$$

Utilizing recurrence relation (6.4) and transformed initial condition(6.5), we obtain

$$\begin{aligned} Y_{q,\alpha,\beta}(1) &= \lambda \frac{1}{\Gamma_q(\alpha - (1-\alpha)(1-\beta) + 1)} y_0, \\ Y_{q,\alpha,\beta}(2) &= \lambda^2 \frac{1}{\Gamma_q(2\alpha - (1-\alpha)(1-\beta) + 1)} y_0, \\ Y_{q,\alpha,\beta}(3) &= \lambda^3 \frac{1}{\Gamma_q(3\alpha - (1-\alpha)(1-\beta) + 1)} y_0. \end{aligned} \quad (6.6)$$

Using these values of $Y_{q,\alpha}(k)$ in Eq. (5.2), we get

$$\begin{aligned} y(x) &= y_0 x^{-(1-\alpha)(1-\beta)} + \lambda y_0 \frac{1}{\Gamma_q(\alpha - (1-\alpha)(1-\beta) + 1)} x^{\alpha - (1-\alpha)(1-\beta)} \\ &\quad + \lambda^2 y_0 \frac{1}{\Gamma_q(2\alpha - (1-\alpha)(1-\beta) + 1)} x^{2\alpha - (1-\alpha)(1-\beta)} + \dots, \\ y(x) &= y_0 x^{-(1-\alpha)(1-\beta)} \left[1 + \lambda \frac{1}{\Gamma_q(\alpha + \alpha + \beta(1-\alpha))} x^\alpha \right. \\ &\quad \left. + \lambda^2 \frac{1}{\Gamma_q(2\alpha + \alpha + \beta(1-\alpha))} x^{2\alpha} + \dots \right]. \end{aligned}$$

In view of the q -Mittag-Leffler function [3] defined as

$$e_{\alpha,\beta}(x; q) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma_q(\alpha n + \beta)}, \quad (|x(1-q)^\alpha| < 1), \quad (6.7)$$

we arrive at (6.3). $\square$

Problem 2 For $y \in L_q^1[0, a]$, $I_q^{(1-\beta)(1-\alpha)} y \in \mathcal{AC}_q[0, a]$, $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 < |q| < 1$, let us consider the following q -initial value problem

$$D_q^{\alpha,\beta} y(x) - \lambda y(x) = x^{n\alpha - (1-\alpha)(1-\beta)}, \quad (6.8)$$

where λ is a real number with initial condition

$$I_q^{(1-\beta)(1-\alpha)} y(x)|_{x \rightarrow 0^+} = y_0. \quad (6.9)$$

The solution of the above problem is given by

$$\begin{aligned} y(x) &= y_0 x^{-(1-\alpha)(1-\beta)} e_{\alpha,\alpha+\beta(1-\alpha)}(\lambda x^\alpha; q) \\ &\quad + \frac{\Gamma_q(n\alpha - (1-\alpha)(1-\beta) + 1)}{\Gamma_q((n+1)\alpha - (1-\alpha)(1-\beta) + 1)} x^{(n+1)\alpha - (1-\alpha)(1-\beta)}. \end{aligned} \quad (6.10)$$

Proof Taking generalized q -differential transform (5.1) of (6.8) and initial condition (6.9), using properties of generalized q -differential transform given in Theorem 5.1, we get

$$\frac{\Gamma_q(\alpha(k+1) - (1-\alpha)(1-\beta) + 1)}{\Gamma_q(\alpha k - (1-\alpha)(1-\beta) + 1)} Y_{q,\alpha,\beta}(k+1) - \lambda Y_{q,\alpha,\beta}(k) = \delta(k-n) \quad (6.11)$$

and

$$Y_{q,\alpha,\beta}(0) = \frac{y_0}{\Gamma_q(1 - (1-\alpha)(1-\beta))}. \quad (6.12)$$

Utilizing recurrence relation (6.11) and transformed initial condition (6.12), we obtain

$$\begin{aligned}
 Y_{q,\alpha,\beta}(1) &= \lambda \frac{1}{\Gamma_q(\alpha - (1 - \alpha)(1 - \beta) + 1)} y_0 \\
 Y_{q,\alpha,\beta}(2) &= \lambda^2 \frac{1}{\Gamma_q(2\alpha - (1 - \alpha)(1 - \beta) + 1)} y_0 \\
 Y_{q,\alpha,\beta}(3) &= \lambda^3 \frac{1}{\Gamma_q(3\alpha - (1 - \alpha)(1 - \beta) + 1)} y_0 \\
 &\vdots \\
 Y_{q,\alpha,\beta}(n+1) &= \lambda^{n+1} \frac{1}{\Gamma_q((n+1)\alpha - (1 - \alpha)(1 - \beta) + 1)} y_0 \\
 &\quad + \frac{\Gamma_q(n\alpha - (1 - \alpha)(1 - \beta) + 1)}{\Gamma_q((n+1)\alpha - (1 - \alpha)(1 - \beta) + 1)}.
 \end{aligned}$$

Using these values of $Y_{q,\alpha}(k)$ in Eq. (5.2)

$$\begin{aligned}
 y(x) &= y_0 x^{-(1-\alpha)(1-\beta)} \left[1 + \lambda \frac{1}{\Gamma_q(\alpha + \alpha + \beta(1 - \alpha))} x^\alpha \right. \\
 &\quad \left. + \lambda^2 \frac{1}{\Gamma_q(2\alpha + \alpha + \beta(1 - \alpha))} x^{2\alpha} + \dots \right] \\
 &\quad + \frac{\Gamma_q(n\alpha - (1 - \alpha)(1 - \beta) + 1)}{\Gamma_q((n+1)\alpha - (1 - \alpha)(1 - \beta) + 1)} x^{(n+1)\alpha - (1-\alpha)(1-\beta)}.
 \end{aligned}$$

In view of the q -Mittag-Leffler function defined by (6.7), we arrive at (6.10). $\square$

7 Conclusion

In this paper, we have defined composite fractional q -derivative which is a q -extension of generalized Riemann Liouville fractional derivative defined by Hilfer [6]. It also provides generalization of both Riemann–Liouville type fractional q -derivative and Caputo type fractional q -derivative. We study some basic properties of this q -derivative. Next, we have established generalized q -Taylor's formula involving composite fractional q -derivatives. We have also defined generalized q -differential transform and solved two fractional q -difference equations using this method.

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