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On the Stability of Thin-Walled Circular Cylindrical Shells under Static and Periodic Radial Loading

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Abstract

In this paper, we study the dynamic behavior of the cylindrical shells subjected to static and time-varying (periodic) radial loading. Flügge-Luré-Byrne shell theory has been used to derive dynamic equations in the form of nonlinear partial differential equations (PDEs). Linearization of PDEs about the steady-state periodic solutions, followed by Galerkin projection, yields a system of ordinary differential equations (ODEs) with time-varying periodic coefficients. Those are similar to the higher-order Mathieu-Hill equations and their stability analyses have been ascertained using Floquet theory. Dynamic instabilities of the cylindrical shells have been presented through stability charts in the plane of forcing parameters, namely, the static component of the pressure, the

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amplitude, and frequency of the fluctuating component of the pressure. Also, possible sub and super harmonic resonances have been explored through the stability charts. Four different boundary conditions, viz., simply supported, clamped-free, pinned-free (axial constraint at the pinned end) and pinned-free (axial constraint at the free end) have been considered in the present analysis. Presented charts can be very useful during the design of the shell structures subjected to static/dynamic loading.

Keywords: Cylindrical Shells, Buckling, Parametric Instabilities, Floquet Theory, Mathieu-Hill Equations

Nomenclature

$x, \theta, \rho(= R + z), t$: axial, tangential, radial and time coordinates

z : distance of a point from mid surface along the radial direction

R, h, L : radius of the mid surface, thickness and length of cylinder

u_1, u_2, u_3 : displacement components of a generic point along the axial, circumferential and radial directions, respectively

u, v, w : displacement components of a point on the middle surface along the axial, circumferential and radial directions, respectively

E, ν, ρ_m : Young's modulus, Poisson's ratio and density of the material

$\sigma_{xx}, \sigma_{\theta\theta}, \sigma_{zz}$: normal stress components

$\sigma_{x\theta}, \sigma_{xz}, \sigma_{z\theta}$: shear stress components

$N_x, N_\theta, N_{x\theta}, N_{\theta x}, Q_x, Q_\theta$: stress resultants (force per unit length)

$M_x, M_\theta, M_{x\theta}, M_{\theta x}$: moment resultants (moment per unit length)

n : circumferential wavenumber

m : axial half wavenumber

$P (= F_0 + F_1 \cos(\omega t))$: applied radial pressure

ω : excitation frequency

k : a positive integer i.e. $k = 1, 2, 3, \dots$

1 Introduction

1.1 Background

Thin-walled structures have been widely applied in the present developing world. Aerospace, civil constructions, marine and oil industry, etc. are mainly important areas where thin-walled structures are used on a big scale. As we know that many diverse composite materials are used in plate and shell structures, but there are still many unsolved problems related to homogeneous structures. These problems should be firstly solved before application of composite materials and study their effect on mechanical response. One of these problems is buckling phenomena of the thin-walled circular cylindrical shells subjected to static and/or periodic radial uniform pressures. The dynamic buckling or stability analysis of the thin cylinder under static and/or time-varying radial loading is very important from a safety point of view, especially for reactors, marine, and aerospace structures where external loading is not constant with time. Dynamic buckling of thin cylindrical shells occurs due to variable axial, as well as radial loading that can be caused by axial impact, carrying a top mass with base excitation, fluctuations in the pressure surrounding the shell structures.

1.2 Literature survey

Kovtunov [1] has obtained the dynamic instability zones for the circular cylindrical shells subjected to a time-varying axial load. He applied Bubnov-Galerkin projection [2] to the equations of motion for the cylinder and obtained ordinary differential equations (ODEs) with time-varying coefficients. Wei *et al.* [3] have studied experimentally and theoretically the dynamic buckling of a circular cylindrical shell under axial impact. The authors have also investigated the buckling behavior of a circular cylindrical shell, containing initial imperfections, under relatively low impact speed. Mallon *et al.* [4] presented the dynamic buckling analysis of a thin cylindrical vertical shell carrying mass at the top with harmonic base excitation. They have employed Donnell's nonlinear shell theory to derive a semi-analytical model, which exactly satisfies the boundary conditions. The obtained results are compared with FEA results. Huyan and Smitjes [5] have calculated the dynamic critical load of the imperfect cylindrical shell subjected to sudden axial compression. The finite element method has been employed to find the dynamic response of the cylinder. They have found that the

dynamic critical load is significantly influenced by load duration and imperfection. Dynamic critical load decreases with increasing load duration and imperfection magnitude. Pellicano [6] has performed a theoretical and experimental study on a circular cylindrical shell carrying a top mass under base excitation. Experiments showed the emergence of strong non-linearity, when the first axisymmetric mode is excited and it leads to the large amplitude of vibration for a wide range of frequencies. The theoretical model also shows very good agreement with experiments. Chitra and Priyadarsini [7] have investigated the dynamic buckling behavior of the composite cylindrical shell under axial impulse. Dynamic loads are found larger than the static critical load when the time of application is less than the natural period of the cylinder. In the study of dynamic buckling for the cylindrical shells under axial impact, Xu *et al.* [8] have observed axisymmetric and non-axisymmetric buckling.

A few studies are available on the stability analysis of cylindrical shells under varying radial loads [9–11]. Varying radial loads include wind pressure, varying radial load with time, fluctuating hydrostatic pressure in case of the submerged cylinder in liquid. Wah [10] has performed the dynamic buckling analysis of a thin ring (infinitely long cylindrical shell was considered as a collection of rings) subjected to impulse load and investigated the behavior of the ring in five distinct regimes of parameters. Sokolov [12] has investigated the response of cylindrical shells under varying lateral pressure using Flügge shell theory. However, he did not discuss the stability of cylindrical shells under varying lateral pressure. Kapania and Yang [13] performed a stability analysis of cylindrical shells under wind pressure using the finite element method. Subsequently, the effect of wind and snowfall together were incorporated by Kapania [14]. Mustafa and his coworkers [15] studied, via finite element analysis (FEA), the dynamic buckling of a cylindrical shell submerged in a liquid and in air with impulse pressure. Aksogan [16] has studied, using Donnell's theory, the dynamic buckling of a cylindrical shell with variable thickness and under uniform pressure which is a power function of time. Jiashen and Jibin [17] and Virella *et al.* [18] have studied the instability of liquid storage cylindrical tanks subjected to horizontal seismic force excitation. Kumar *et al.* [19] have studied the parametric instabilities of infinitely long cylindrical shells under radially fluctuating pressure. They have presented stability charts in the planes of different forcing parameters (e.g., forcing frequency, the amplitude of fluctuating part, and the static part of pressure) for infinitely long circular cylindrical shells.

1.3 Novelty of the paper

As we observe, there are very limited literature available on the dynamic buckling of cylindrical shells under radial loading. Present work is an attempt to analyse the parametric instabilities of the thin-walled circular cylindrical shells under static and periodic radial loading.

This work is based on formulation developed by Kumar et al. [19] (first author and his collaborators), where they have identified two new terms (linear and nonlinear) appeared together in equation of motion along circumferential direction. These two terms have been missing in many existing shell theories. They have shown the order of each term to justify retaining and neglecting any term (linear and/or nonlinear) in shell dynamic equations. Further that developed model has been reduced to study the parametric instability of the infinitely long cylindrical shell under dynamic radial loading. The present study use full model to investigate the stability analysis of the cylindrical shells with finite length and general BCs. Yao [9] has performed the dynamic of the simply supported cylindrical shells under periodic axial and radial loading. In this analysis, he has made several assumptions in his formulation e.g., hoop stress is independent of the spacial coordinate such assumption is valid only for long cylindrical shells ($L/R \geq 4$) and lower wavenumbers (axial and circumferential). He has also assumed zero deflections under pre-stress condition and neglected in-plane inertia terms. As per Yao's model, radial pressure does not affect beam mode, but first author has shown in his previous work (Kumar et al. [20]) that radial pressure has significant effect on the beam mode too. Stability charts presented in the plane of two parameters which are functions of geometric and forcing parameters. Thus it is very difficult extract information directly from the stability chart in context of stability of the cylinder, e.g., exact static critical pressure, resonance conditions (primary, principle parametric and super-harmonic), variations of critical radial pressure and natural frequencies with axial and circumferential wavenumbers etc. Similarly Lindberg and Florence [11] have discussed the dynamic buckling of ring and long cylindrical shells under radially fluctuating pressure, where they have assumed long cylinders as collection of rings.

Through the present work, we attempt to fill these gaps and present a comprehensive study on static and dynamic instabilities of the cylindrical shells for general boundary conditions. In this work, full dynamic equations along axial, circumferential and radial directions are considered with incorporating assumptions consistently. A detail procedure has been outlined

to formulate higher order Mathieu-hill equations for the cylindrical shells subjected to periodic radial loading. Present model is valid for all range of L/R ratios, wavenumbers and boundary conditions. For static case, we perform theoretical analysis and verify with FEA results using ABAQUS but for dynamic case only numerical studies are presented. Results for the dynamic instabilities of the cylindrical shells are presented through stability charts in the plane of forcing parameters, namely, the static component of the pressure, the amplitude, and frequency of the fluctuating component of the pressure. Thus the present work gives a complete picture of the instabilities of the cylindrical shells under static and periodic radial loading for any combination of the forcing parameters.

The paper is organized as follows. In Section 2, we present basic formulations for different types of loading and boundary conditions (BCs). In Section 3, we first present results for the static buckling, and, subsequently, the stability charts for thin-walled circular cylindrical shells under dynamic loading are presented for four different boundary conditions. Finally, we conclude in Section 4.

2 Basic Formulation

We consider a thin-walled circular cylindrical shell with radius R and thickness h as shown in Fig. 1. Here u_1 , u_2 , and u_3 are displacement components of a generic point with coordinate $(x, \theta, \rho (= R + z))$ along axial, tangential and radial directions, respectively. In the present analysis, we start with the model developed by Kumar et al [19] which is based on Flügge-Lur -Byrne shell theory [21]. The displacement field (i.e. relation between the displacement components of a generic point (u_1, u_2, u_3) and the displacement components of a point on the mid-surface (u, v, w)) for Fl -Lur -Byrne theory are

$$u_1(x, \theta, z, t) = u(x, \theta, t) - z \frac{\partial}{\partial x} w(x, \theta, t) \quad (1a)$$

$$u_2(x, \theta, z, t) = v(x, \theta, t) - \frac{z}{R} \left(\frac{\partial}{\partial \theta} w(x, \theta, t) - v(x, \theta, t) \right), \quad (1b)$$

$$u_3(x, \theta, z, t) = w(x, \theta, t) \quad (1c)$$

where u, v, w are displacement components of a point on the mid-surface along axial, tangential and radial directions, respectively.

The dynamic equations of motion (developed by Kumar *et al.* [19]) in terms of stress and

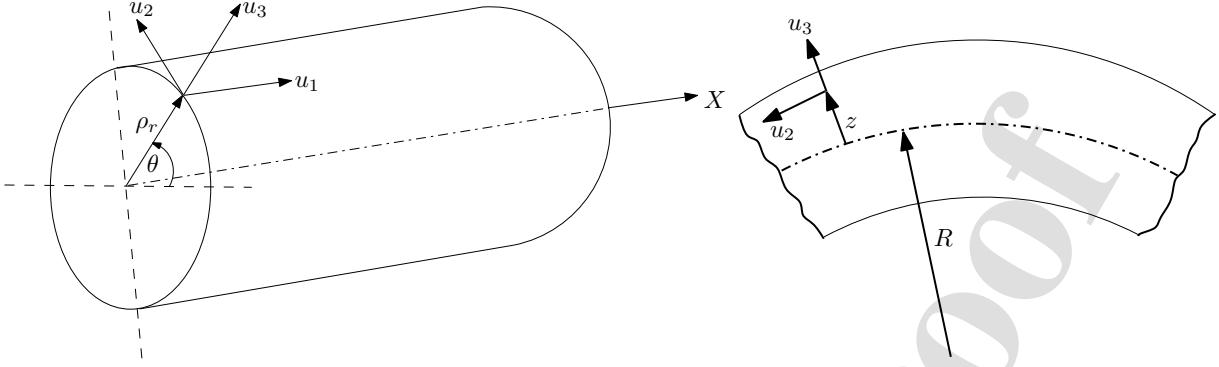


Figure 1: Schematic of a cylindrical shell showing the displacements of a generic point in the shell with cylindrical coordinates $(x, \theta, \rho(= R + z))$.

moment resultants along axial, tangential and radial directions, respectively, are

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{x\theta}}{R\partial\theta} - ch\dot{u}(x, \theta, t) - \rho_m h \ddot{u}(x, \theta, t) = 0, \quad (2a)$$

$$\frac{\partial N_{x\theta}}{\partial x} + \frac{\partial N_\theta}{R\partial\theta} + \frac{Q_\theta}{R} + \frac{N_{x\theta}}{R} \frac{\partial w(x, \theta, t)}{\partial x} + \frac{N_\theta}{R^2} \frac{\partial w(x, \theta, t)}{\partial\theta} - ch\dot{v}(x, \theta, t) - \rho_m h \ddot{v}(x, \theta, t) = 0, \quad (2b)$$

$$\begin{aligned} & \frac{\partial Q_x}{\partial x} + \frac{\partial Q_\theta}{R\partial\theta} + N_x \frac{\partial^2 w(x, \theta, t)}{\partial x^2} + N_{x\theta} \frac{\partial^2 w(x, \theta, t)}{\partial x R \partial\theta} + N_{\theta x} \frac{\partial^2 w(x, \theta, t)}{\partial x R \partial\theta} + N_\theta \frac{\partial^2 w(x, \theta, t)}{R^2 \partial\theta^2} - \frac{N_\theta}{R} + \\ & \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{\theta x}}{R\partial\theta} \right) \frac{\partial w(x, \theta, t)}{\partial x} + \left(\frac{\partial N_{x\theta}}{\partial x} + \frac{\partial N_\theta}{R\partial\theta} \right) \frac{\partial w(x, \theta, t)}{R\partial\theta} + P - ch\dot{w}(x, \theta, t) - \rho_m h \ddot{w}(x, \theta, t) = 0, \end{aligned} \quad (2c)$$

where ρ_m and c is the density of the cylinder material and coefficient of the linear viscous damping. Over-dot in Eqs. (2a) through (2c) represents derivative with respect to time. The time varying radial load P takes the form $P = F_0 + F_1 \cos \omega t$, where ω is forcing frequency in (rad/sec), F_0 and F_1 are the static and amplitude components of the time varying load. Definitions of the resultant stress resultants N_x , N_θ , $N_{x\theta}$, $N_{\theta x}$, Q_x , Q_θ and moment resultants M_x , M_θ , $M_{x\theta}$ and $M_{\theta x}$ are given in [19] (as shown in Fig. 2). The resultant stresses per unit length of the undeformed middle surface are defined as

$$N_x = \int_{-h/2}^{h/2} \sigma_{xx} \left(1 + \frac{z}{R} \right) dz, \quad N_\theta = \int_{-h/2}^{h/2} \sigma_{\theta\theta} dz, \quad N_{x\theta} = \int_{-h/2}^{h/2} \sigma_{x\theta} \left(1 + \frac{z}{R} \right) dz, \quad (3a)$$

$$Q_x = \int_{-h/2}^{h/2} \sigma_{xz} \left(1 + \frac{z}{R} \right) dz, \quad Q_\theta = \int_{-h/2}^{h/2} \sigma_{\theta z} dz, \quad (3b)$$

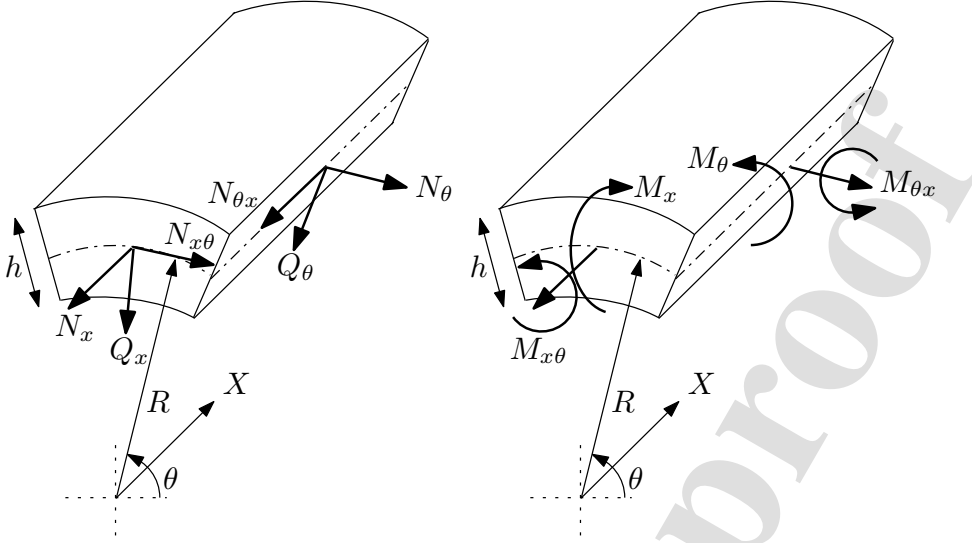


Figure 2: Resultant stresses and moments per unit length of middle surface.

and the resultant moments per unit length are

$$M_x = \int_{-h/2}^{h/2} \sigma_{xx} \left(1 + \frac{z}{R}\right) z dz, \quad M_\theta = \int_{-h/2}^{h/2} \sigma_{\theta\theta} z dz, \quad M_{x\theta} = \int_{-h/2}^{h/2} \sigma_{x\theta} \left(1 + \frac{z}{R}\right) z dz. \quad (4)$$

where σ_{xx} , $\sigma_{\theta\theta}$ are normal stress components and $\sigma_{x\theta}$, $\sigma_{\theta x}$, σ_{xz} , $\sigma_{\theta z}$, σ_{zx} , $\sigma_{z\theta}$ are shear components of the stress (see [19]). The resultant transverse shear stresses Q_x and Q_θ are related to the resultant moments through the relations

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{\theta x}}{R \partial \theta}, \quad \text{and} \quad Q_\theta = \frac{\partial M_{x\theta}}{\partial x} + \frac{\partial M_\theta}{R \partial \theta}. \quad (5)$$

Note that we neglect the rotary inertia terms $\left(\frac{\rho_m h^3}{12} \frac{\partial^3 w}{\partial x \partial t^2}, \quad \text{and} \quad \frac{\rho_m h^3}{12} \frac{\partial^3 w}{\partial \theta \partial t^2}\right)$ in Eq. (5). These terms lead to a few additional rotary inertia terms $\left(\frac{\rho_m h^3}{12} \frac{\partial^3 w}{\partial x^2 \partial t^2}, \quad \text{and} \quad \frac{\rho_m h^3}{12} \frac{\partial^3 w}{\partial \theta^2 \partial t^2}\right)$ in Eqs. (2b) and (2c). An order of magnitude analysis reveals that these terms involving the rotary inertia are $\mathcal{O}(h^3)$, whereas the translational inertia is $\mathcal{O}(h)$. Since, the present analysis pertains to lower axial and circumferential wavenumbers, we neglect the rotary inertia terms in the current work (see [20]). The expressions for the stress and moment resultants (using

Flügge-Lur -Byrne shell theory) are (for detail see [19, 20]) given in Eqs. (6a)-(6j).

$$N_x = \frac{Eh}{1-\nu^2} \left[\frac{\partial u}{\partial x} + \nu \left(\frac{\partial v}{R\partial\theta} + \frac{w}{R} \right) \right] + \frac{Eh^3}{12R(1-\nu^2)} \left[-\frac{\partial^2 w}{\partial x^2} + \nu \left(-\frac{\partial^2 w}{R^2\partial\theta^2} - \frac{w}{R^2} \right) \right], \quad (6a)$$

$$N_\theta = \frac{Eh}{1-\nu^2} \left[\frac{\partial v}{R\partial\theta} + \frac{w}{R} + \nu \frac{\partial u}{\partial x} \right], \quad (6b)$$

$$N_{x\theta} = \frac{Eh}{2(1+\nu)} \left[\frac{\partial v}{\partial x} + \frac{\partial u}{R\partial\theta} \right] + \frac{Eh^3}{24R(1+\nu)} \left[-2\frac{\partial^2 w}{R\partial x\partial\theta} + \frac{\partial v}{R\partial x} - \frac{\partial u}{R^2\partial\theta} \right], \quad (6c)$$

$$N_{\theta x} = \frac{Eh}{2(1+\nu)} \left[\frac{\partial v}{\partial x} + \frac{\partial u}{R\partial\theta} \right], \quad (6d)$$

$$M_x = \frac{Eh^2}{12(1-\nu^2)} \left[-\frac{\partial^2 w}{\partial x^2} - \nu \frac{\partial^2 w}{R^2\partial\theta^2} + \frac{1}{R} \frac{\partial u}{\partial x} + \nu \frac{\partial v}{R^2\partial\theta} \right] \quad (6e)$$

$$M_\theta = \frac{Eh^3}{12(1-\nu^2)} \left[-\frac{\partial^2 w}{R^2\partial\theta^2} - \frac{w}{R^2} - \nu \frac{\partial^2 w}{\partial x^2} \right], \quad (6f)$$

$$M_{x\theta} = \frac{Eh^3}{12(1+\nu)} \left[\frac{1}{R} \frac{\partial v}{\partial x} - \frac{\partial^2 w}{R\partial\theta\partial x} \right], \quad (6g)$$

$$M_{\theta x} = \frac{Eh^3}{24(1+\nu)} \left[-2\frac{\partial^2 w}{R\partial\theta\partial x} + \frac{1}{R} \frac{\partial v}{\partial x} - \frac{\partial u}{R\partial\theta} \right], \quad (6h)$$

$$Q_x = \frac{Eh^3}{12(1-\nu^2)} \left[\frac{\partial^2 u}{R\partial x^2} + \nu \left(\frac{1}{R^2} \frac{\partial^2 v}{\partial x\partial\theta} + \frac{\partial w}{R\partial x} \right) - \frac{\partial^3 w}{\partial x^3} + \nu \left(-\frac{1}{R^2} \frac{\partial^3 w}{\partial x\partial\theta^3} - \frac{\partial w}{R^2\partial x} \right) \right] \\ + \frac{Eh^3}{24R(1+\nu)} \left[\frac{2}{R} \frac{\partial^3 w}{\partial x\partial\theta^2} + \frac{\partial^2 v}{R\partial x\partial\theta} - \frac{\partial^2 u}{R^2\partial\theta^2} \right], \quad (6i)$$

$$Q_\theta = \frac{Eh^3}{12R(1-\nu^2)} \left[-\frac{\partial^3 w}{R^2\partial\theta^3} - \frac{\partial w}{R^2\partial\theta} - \nu \frac{\partial^3 w}{\partial x^2\partial\theta} \right] + \frac{Eh^3}{24(1+\nu)} \left[2\frac{\partial^2 v}{R\partial x^2} - 2\frac{\partial^3 w}{R\partial x^2\partial\theta} \right], \quad (6j)$$

Substituting Eqs. (6a)-(6j) into Eqs. (2a)-(2c) yields three dynamic equations of the cylindrical shell in terms of mid-surface displacement components along the three directions.

The cylinder has defined BCs at its ends and given by Eqs. (7)-(10) [20].

BC- A (simply supported ends),

$$at \ x = 0, \begin{cases} w(x) = 0, \\ v(x) = 0, \\ N_x = 0, \\ M_x = 0. \end{cases} \quad \text{and} \quad at \ x = L, \begin{cases} w(x) = 0, \\ v(x) = 0, \\ N_x = 0, \\ M_x = 0. \end{cases} \quad (7)$$

BC- B (pinned-free ends with axial constraint at pinned end),

$$\text{at } x = 0, \begin{cases} w(x) = 0, \\ v(x) = 0, \\ u(x) = 0, \\ M_x = 0. \end{cases} \quad \text{and} \quad \text{at } x = L, \begin{cases} N_x = 0, \\ M_x = 0, \\ Q_x = 0, \\ N_{x\theta} = 0. \end{cases} \quad (8)$$

BC- C (fixed-free ends),

$$\text{at } x = 0, \begin{cases} w(x) = 0, \\ u(x) = 0, \\ v(x) = 0, \\ \frac{\partial w(x)}{\partial x} = 0. \end{cases} \quad \text{and} \quad \text{at } x = L, \begin{cases} M_x = 0, \\ N_x = 0, \\ Q_x = 0, \\ N_{x\theta} = 0. \end{cases} \quad (9)$$

BC-D (pinned and axial constraint at free end),

$$\text{at } x = 0, \begin{cases} w(x) = 0, \\ v(x) = 0, \\ N_x = 0, \\ M_x = 0. \end{cases} \quad \text{and} \quad \text{at } x = L, \begin{cases} M_x = 0, \\ u(x) = 0, \\ Q_x = 0, \\ N_{x\theta} = 0. \end{cases} \quad (10)$$

We follow the procedures as discussed by Kumar *et al.* [19] for an infinitely long cylinder. However, for simplicity, we neglect the damping term for the finite shells. The steady state displacements $u_s(x, t)$, $v_s(x, t)$ and $w_s(x, t)$ along the axial, tangential, and radial directions, respectively, for the cylindrical shells with finite length subjected to the time-varying load $P = F_0 + F_1 \cos(\omega t)$ have been assumed as

$$u_s(x, t) = u_0(x) + u_1(x) \cos(\omega t), \quad v_s(x, t) = 0 \quad \text{and} \quad w_s(x, t) = w_0(x) + w_1(x) \cos(\omega t). \quad (11)$$

The steady state solution along the circumferential direction is zero due to axisymmetric loading. Substituting the expressions given in Eq. (11) into the equations of motion (2a)-(2c) and equating the coefficients of $\cos(\omega t)$ on the left hand side to zero, we obtain a system of ODEs for $u_0(x)$, $u_1(x)$, $w_0(x)$ and $w_1(x)$ (assuming N_x is zero for steady state solutions, since

loading is fully axi-symmetric and ends of the cylinder are free to move along axial direction),

$$\begin{aligned} \frac{Eh^3}{12(1-\nu^2)} \left(\frac{h^2}{12R^2} - 1 \right) \frac{d^4}{dx^4} w_0(x) + \frac{Eh^3\nu}{6R^2(1-\nu^2)} \left(\frac{h^2}{24R^2} - 1 \right) \frac{d^2}{dx^2} w_0(x) \\ + \frac{Eh}{R^2(1-\nu^2)} \left(\nu^2 + \frac{\nu^2 h^2}{12R^2} - 1 \right) w_0(x) + F_0 = 0, \end{aligned} \quad (12)$$

$$\frac{Eh}{(1-\nu^2)} \frac{d^2}{dx^2} u_0(x) = \frac{Eh^3}{12R(1-\nu^2)} \frac{d^3}{dx^3} w_0(x) + \frac{Eh\nu}{R(1-\nu^2)} \left(\frac{h^2}{12R^2} - 1 \right) \frac{d}{dx} w_0(x), \quad (13)$$

$$\begin{aligned} \frac{Eh^3}{12(1-\nu^2)} \left(\frac{h^2}{12R^2} - 1 \right) \frac{d^4}{dx^4} w_1(x) + \frac{Eh^3\nu}{6R^2(1-\nu^2)} \left(\frac{h^2}{24R^2} - 1 \right) \frac{d^2}{dx^2} w_1(x) \\ + \frac{Eh}{R^2(1-\nu^2)} \left(\frac{\rho h \omega^2 R^2 (1-\nu^2)}{Eh} + \nu^2 + \frac{\nu^2 h^2}{12R^2} - 1 \right) w_1(x) + F_1 = 0, \end{aligned} \quad (14)$$

$$\frac{Eh}{(1-\nu^2)} \frac{d^2}{dx^2} u_1(x) + \rho h \omega^2 u_1(x) = \frac{Eh^3}{12R(1-\nu^2)} \frac{d^3}{dx^3} w_1(x) + \frac{Eh\nu}{R(1-\nu^2)} \left(\frac{h^2}{12R^2} - 1 \right) \frac{d}{dx} w_1(x), \quad (15)$$

We consider thin-walled cylindrical shells, i.e., $h/R \ll 1$, hence we can neglect $(h/R)^2$ with respect to unity. Also the terms involve in Eqs. (14) and (15) $\frac{\rho h \omega^2 R^2 (1-\nu^2)}{Eh}$ and $\frac{\rho h \omega^2 (1-\nu^2)}{Eh}$ are order of $\mathcal{O}\left(\frac{h^2}{R^2}\right)$, hence they are also neglected with respect to unity.

Rewrite Eqs. (12)-(15) using above approximations,

$$\frac{Eh^3}{12(1-\nu^2)} \frac{d^4}{dx^4} w_0(x) + \frac{Eh^3\nu}{6R^2(1-\nu^2)} \frac{d^2}{dx^2} w_0(x) + \frac{Eh}{R^2} w_0(x) - F_0 = 0, \quad (16)$$

$$\frac{Eh}{(1-\nu^2)} \frac{d^2}{dx^2} u_0(x) = \frac{Eh^3}{12R(1-\nu^2)} \frac{d^3}{dx^3} w_0(x) - \frac{Eh\nu}{R(1-\nu^2)} \frac{d}{dx} w_0(x), \quad (17)$$

$$\frac{Eh^3}{12(1-\nu^2)} \frac{d^4}{dx^4} w_1(x) + \frac{Eh^3\nu}{6R^2(1-\nu^2)} \frac{d^2}{dx^2} w_1(x) + \frac{Eh}{R^2} w_1(x) - F_1 = 0, \quad (18)$$

$$\frac{Eh}{(1-\nu^2)} \frac{d^2}{dx^2} u_1(x) = \frac{Eh^3}{12R(1-\nu^2)} \frac{d^3}{dx^3} w_1(x) - \frac{Eh\nu}{R(1-\nu^2)} \frac{d}{dx} w_1(x), \quad (19)$$

From Eqs. (16)-(19), we observe that the form of steady state solutions for $w_0(x)$ and $w_1(x)$ and $u_0(x)$ and $u_1(x)$ are the same and look like as in the case of the cylindrical shells under static radial pressure F_0 and F_1 , respectively. We rewrite Eqs.(16),

$$\frac{d^4}{dx^4} w_0(x) + a_1 \frac{d^2}{dx^2} w_0(x) + b_1 w_0(x) + d_1 = 0, \quad (20)$$

where $a_1 = \frac{\nu}{2R^2}$, $b_1 = \frac{12(1-\nu^2)}{R^2 h^2}$ and $d_1 = -\frac{12(1-\nu^2)F_0}{Eh^3}$. Equation (20) is a 4th order ordinary differential equation and closed-form solution is given as

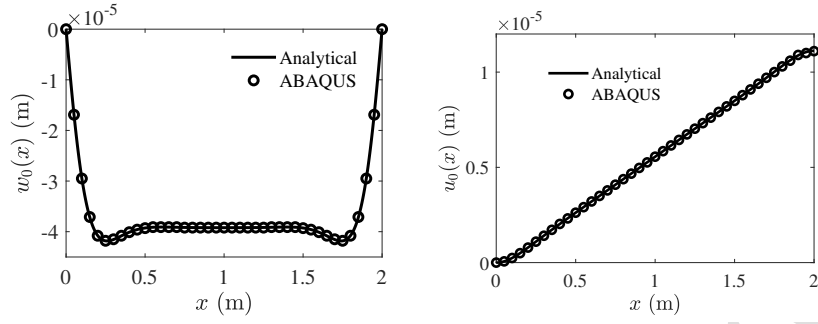
$$w_0(x) = -\frac{d_1}{b_1} + c_1 \exp(-\lambda_1 x) + c_2 \exp(\lambda_1 x) + c_3 \exp(-\lambda_2 x) + c_4 \exp(\lambda_2 x), \quad (21)$$

where $\lambda_1 = \frac{\sqrt{-2\sqrt{a_1^2 - 4b_1} - 2a_1}}{2}$ and $\lambda_2 = \frac{\sqrt{2\sqrt{a_1^2 - 4b_1} - 2a_1}}{2}$. Using solutions for $w_0(x)$ and $w_1(x)$, we solve Eqs. (17) for $u_0(x)$

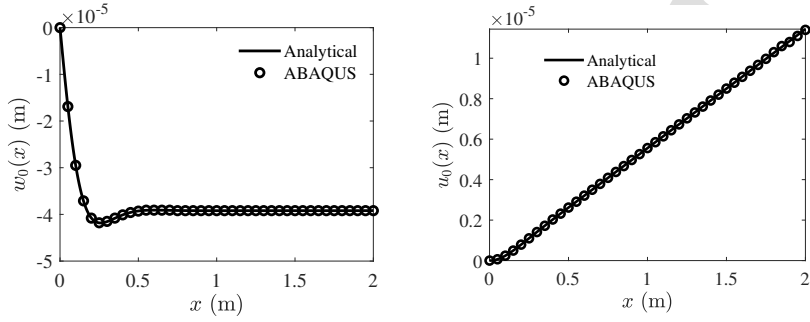
$$u_0(x) = \int \left[\int \left(\frac{h^2}{12R^2} \frac{d^3}{dx^3} w_0(x) - \frac{\nu}{R} \frac{d}{dx} w_0(x) \right) dx \right] dx + c_5 x + c_6, \quad (22)$$

Integration constants ($c_1, c_2, c_3, c_4, c_5, c_6$) are obtained using BCs given in Eqs. (7) through (10). Similarly, we get the closed-form solutions for $w_1(x)$ and $u_1(x)$.

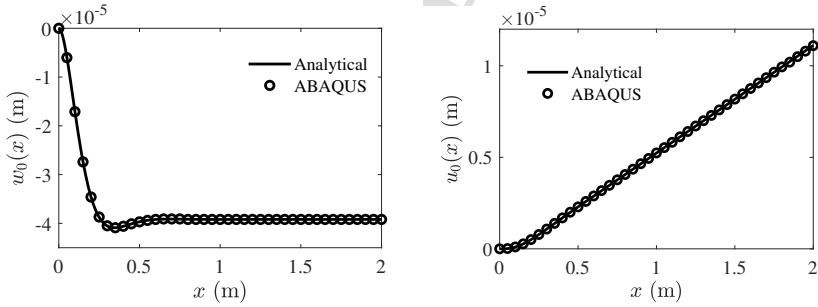
We verify the analytical expressions of the steady state solutions with FEA results using ABAQUS. In all the FE analyses presented here we consider eight noded quadratic shell elements with reduced order integration (S8R) and enhanced hourglass effect. The number of elements was chosen in each case after testing for convergence. The Lanczos method (available with ABAQUS) has been used to solve the linear algebraic equations with a numerical tolerance of 10^{-6} . We have also used subspace method (also available with ABAQUS) and observed that the results obtained from both the Lanczos and subspace methods are close. For $F_1 = 0$, steady state solutions become $u_s(x, t) = u_0(x)$ and $w_s(x, t) = w_0(x)$. Steady state solutions along axial $u_0(x)$ (see right column) and radial $w_0(x)$ (see left column) directions are shown in Fig. 3. We find a very good agreement between the analytical expressions of the steady state solutions and the steady state solutions obtained from FEA using ABAQUS, as shown in Fig. 3 for BC-A through BC-D. It has already been observed from Eqs. (16)-(19), the form of the solutions for $w_0(x)$ and $w_1(x)$, and $u_0(x)$ and $u_1(x)$ are same. So we can easily find the steady state solutions for a cylindrical shell subjected to periodic radial pressure and those have a very good agreement with FEA results using ABAQUS.



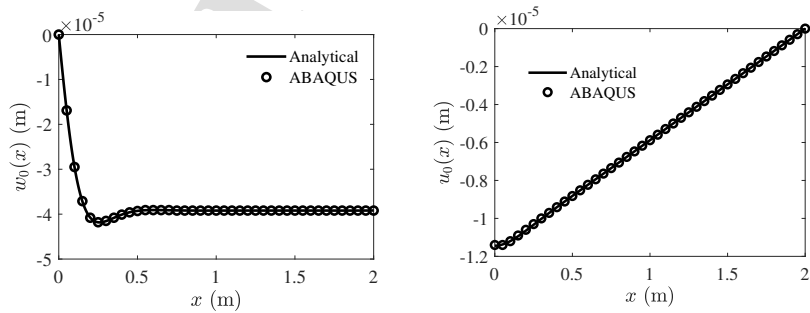
(a) BC-A



(b) BC-B



(c) BC-D



(d)

Figure 3: Verification of the steady state solutions obtained analytically (solving Eqs. (16)-(19)) with FEA results using ABAQUS. Parameters are $E = 2 \times 10^{11}$ N/m², $\rho_m = 7680$ Kg/m³, $\nu = 0.3$, $L = 2$ m, $R = 2$ m, $h = 0.01$ m, $F_1 = 0$, and $F_0 = 2 \times 10^4$ N/m² (radially inward).

Linearization of Eqs. (2a)-(2c) about the steady state solution yields linear PDEs for the respective BCs. Finally, Galerkin projection has been applied to the linearized PDEs with approximate shape functions given by Kumar *et al.* [20]

$$\begin{aligned}\tilde{u} &= U_{mn}(t) \cos\left(\frac{\beta_m}{L}x\right) \cos(n\theta), \\ \tilde{v} &= V_{mn}(t) \sin\left(\frac{\beta_m}{L}x\right) \sin(n\theta), \\ \tilde{w} &= W_{mn}(t) \sin\left(\frac{\beta_m}{L}x\right) \cos(n\theta),\end{aligned}\tag{23}$$

where m is the axial half wavenumber, n is the circumferential wavenumber and $U_{mn}(t)$, $V_{mn}(t)$ and $W_{mn}(t)$ are time varying generalized coordinates along the axial, circumferential and radial directions, respectively. The quantity β_m is different for different boundary conditions and taken from the characteristic equations of Euler-Bernoulli beam for the respective boundary conditions. They are given, read

$$\begin{aligned}\beta_m &= m\pi, & \text{for BC-A} \\ \beta_m &= (2m-1)\pi/2, & \text{for BC-D and} \\ \beta_1 &= 1.875, \beta_2 = 4.694, \beta_3 = 7.857 \dots & \text{for BC-B and BC-C.}\end{aligned}\tag{24}$$

After Galerkin projection, we obtain a system of ODEs

$$\begin{bmatrix} \ddot{U}_{mn}(t) \\ \ddot{V}_{mn}(t) \\ \ddot{W}_{mn}(t) \end{bmatrix} = [\mathbb{A}(t)] \begin{bmatrix} U_{mn}(t) \\ V_{mn}(t) \\ W_{mn}(t) \end{bmatrix},\tag{25}$$

where $[\mathbb{A}(t)]$ is a 3×3 matrix. Expressions of all elements of the matrix $[\mathbb{A}]$ in Eq. (25) are

$$\begin{aligned}\mathbb{A}(1,1) &= \frac{E\beta_m n}{2LR\rho(1-\nu)}, & \mathbb{A}(1,2) &= \frac{E\beta_m^2}{L^2\rho(\nu^2-1)} - \frac{En^2}{2R^2\rho(1+\nu)}, \\ \mathbb{A}(1,3) &= \frac{Eh^2\beta_m\nu(n^2-1)}{12LR^2\rho(1-\nu^2)} - \frac{E\beta_m(\beta_m^2 h^2 + 12L^2\nu)}{12L^3\rho(1-\nu^2)}, \\ \mathbb{A}(2,1) &= \frac{E\beta_m^2 n(h^2(1-\nu) - 12R^2(1+\nu))}{24LR^3\rho(\nu^2-1)} - \frac{E\beta_m n}{LR^2\rho(1+\nu)(\sin(2\beta_m) - 2\beta_m)} \int_0^L \sin\left(\frac{2\beta_m}{L}x\right) \frac{\partial}{\partial x} w_s(x,t) dx, \\ \mathbb{A}(2,2) &= \frac{E\beta_m^2(1-\nu)(4R^2 + h^2) - 8EL^2n^2}{4L^2R^2\rho(\nu^2-1)} - \frac{E\beta_m^2}{L^2R(1+\nu)(\sin(2\beta_m) - 2\beta_m)} \int_0^L \sin\left(\frac{2\beta_m}{L}x\right) \frac{\partial}{\partial x} w_s(x,t) dx, \\ \mathbb{A}(2,3) &= \frac{En(R^2\beta_m^2 h^2(2-\nu) + L^2h^2(n^2-1) + 12L^2R^2)}{12L^2R^4\rho(\nu^2-1)} - \\ &\quad \frac{4E\beta_m n\nu}{R^2\rho(\nu^2-1)(\sin(2\beta_m) - 2\beta_m)} \left[\nu \int_0^L \sin^2\left(\frac{\beta_m}{L}x\right) \frac{\partial}{\partial x} u_s(x,t) dx + \int_0^L \sin\left(\frac{2\beta_m}{L}x\right) w_s(x,t) dx \right],\end{aligned}$$

$$\begin{aligned} \mathbb{A}(3, 1) = & \frac{E\beta_m (L^2 h^2 n^2 (1 - \nu) - 2R^2 (\beta_m^2 h^2 + 12L^2 \nu))}{24L^3 R^3 \rho (\nu^2 - 1)} \\ & - \frac{E (R^3 \beta_m^3 + L^2 R \beta_m n^2 (1 - \nu))}{L^3 R^3 \rho (\nu^2 - 1) (\sin(2\beta_m) - 2\beta_m)} \int_0^L \sin\left(\frac{2\beta_m}{L}x\right) \frac{\partial}{\partial x} w_s(x, t) dx \\ & - \frac{4E\beta_m^3}{L^2 \rho (\nu^2 - 1) (\sin(2\beta_m) - 2\beta_m)} \int_0^L \sin^2\left(\frac{\beta_m}{L}x\right) \frac{\partial^2}{\partial x^2} w_s(x, t) dx, \end{aligned}$$

$$\begin{aligned} \mathbb{A}(3, 2) = & \frac{En (12L^2 + \beta_m^2 h^2 (3 - \nu))}{12L^2 R^2 \rho (\nu^2 - 1)} + \frac{E\beta_m^2 n}{L^2 R \rho (1 + \nu) (\sin(2\beta_m) - 2\beta_m)} \int_0^L \sin\left(\frac{2\beta_m}{L}x\right) \frac{\partial}{\partial x} w_s(x, t) dx \\ & + \frac{4E\beta_m n \nu}{LR \rho (\nu^2 - 1) (\sin(2\beta_m) - 2\beta_m)} \int_0^L \sin^2\left(\frac{\beta_m}{L}x\right) \frac{\partial^2}{\partial x^2} w_s(x, t) dx, \end{aligned}$$

$$\begin{aligned} \mathbb{A}(3, 3) = & \frac{E (L^4 h^2 n^2 (n^2 - 1) + R^2 \beta_m^2 h^2 (R^2 \beta_m^2 + 2L^2 n^2) + 12L^4 R^2)}{12L^4 R^4 \rho (\nu^2 - 1)} \\ & + \frac{E\beta_m (\beta_m^2 h^2 \nu^2 - 12(L^2 n^2 + R^2 \beta_m^2 \nu))}{3L^3 R^3 \rho (\nu^2 - 1) (\sin(2\beta_m) - 2\beta_m)} \int_0^L \sin^2\left(\frac{\beta_m}{L}x\right) w_s(x, t) dx \\ & + \frac{E (R^2 \beta_m^2 (\beta_m^2 h^2 + 24L^2 \nu) + L^2 \beta_m^2 h^2 \nu (n^2 - 2))}{6L^4 R^3 \rho (\nu^2 - 1) (\sin(2\beta_m) - 2\beta_m)} \int_0^L \sin\left(\frac{2\beta_m}{L}x\right) \frac{\partial}{\partial x} w_s(x, t) dx \\ & + \frac{E\beta_m (L^2 h^2 \nu (n^2 - 1) + 2R^2 (\beta_m^2 h^2 + 6L^2 \nu))}{3L^3 R^3 \rho (\nu^2 - 1) (\sin(2\beta_m) - 2\beta_m)} \int_0^L \sin^2\left(\frac{\beta_m}{L}x\right) \frac{\partial^2}{\partial x^2} w_s(x, t) dx \\ & - \frac{Eh^2 \beta_m^2}{6L^2 R \rho (\nu^2 - 1) (\sin(2\beta_m) - 2\beta_m)} \int_0^L \sin\left(\frac{2\beta_m}{L}x\right) \frac{\partial^3}{\partial x^3} w_s(x, t) dx \\ & - \frac{4E\beta_m (L^2 n^2 \nu - R^2 \beta_m^2)}{L^3 R^2 \rho (\nu^2 - 1) (\sin(2\beta_m) - 2\beta_m)} \int_0^L \sin^2\left(\frac{\beta_m}{L}x\right) \frac{\partial}{\partial x} u_s(x, t) dx \\ & + \frac{2E\beta_m^2}{L^2 \rho (\nu^2 - 1) (\sin(2\beta_m) - 2\beta_m)} \int_0^L \sin\left(\frac{2\beta_m}{L}x\right) \frac{\partial^2}{\partial x^2} u_s(x, t) dx. \end{aligned}$$

where $u_s(x, t) = u_0(x) + u_1(x) \cos(\omega t)$ and $w_s(x, t) = w_0(x) + w_1(x) \cos(\omega t)$ are steady solutions (see Appendix A). β_m is defined in Eq. (24) for the respective boundary conditions.

3 Results and Discussion

3.1 Stability under Static Radial Loading

Before presenting the stability charts in the plane of forcing parameters, we calculate the static critical radial pressure P_{cr} for different circumferential and axial half wavenumbers for the cylinder subjected to uniform static radial pressure (i.e., for the case of $F_1 = 0$). Analytical results have been verified with FEA results using ABAQUS. Towards this, first we perform the

convergence studies for $L/R = 5$ (as shown in Fig. 4) and $L/R = 1$ (as shown in Fig. 5). As we increase the number of elements, the FEA results show converging trends with analytical results (as shown in Figs. 4 and 5).

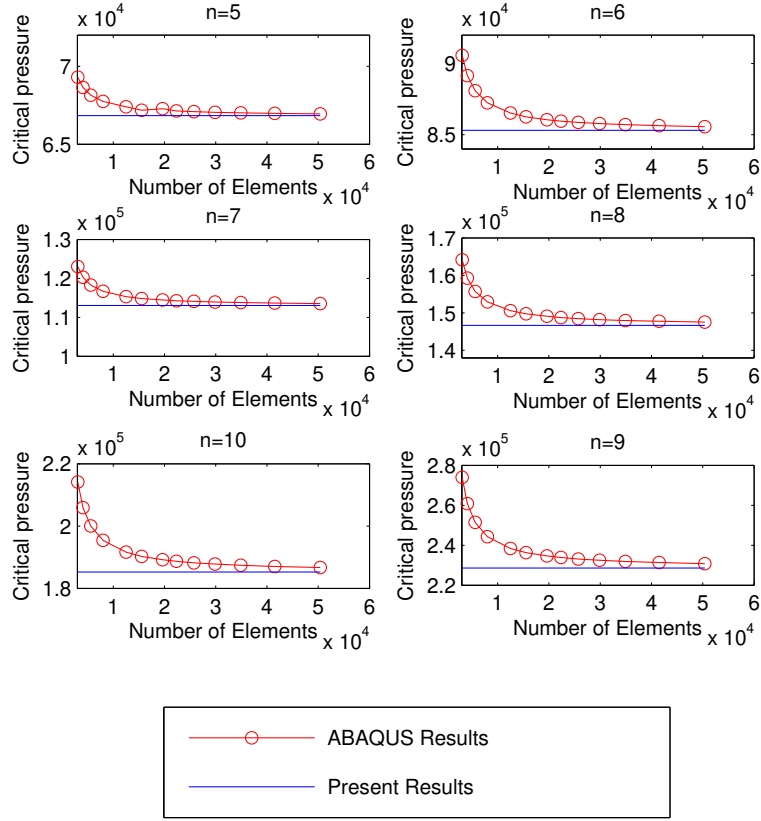


Figure 4: Convergence of results of FEA using ABAQUS with analytical results as number of elements increase for BC-A, using parameters $h = 0.01$ m, $R = 2$ m, $L = 10$ m, $E = 2 \times 10^{11}$ N/m², and $\nu = 0.3$.

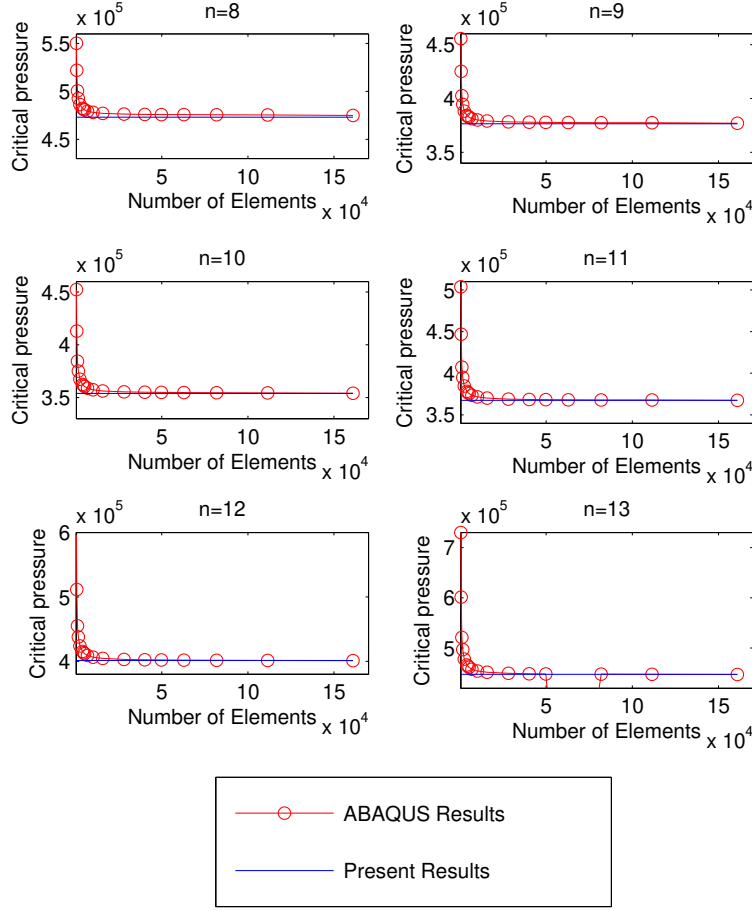


Figure 5: Convergence of results of FEA using ABAQUS with analytical results as number of elements increase, using parameters $h = 0.01$ m, $R = 2$ m, $L = 2$ m, $E = 2 \times 10^{11}$ N/m², and $\nu = 0.3$.

Figure 6 shows the variation of static buckling radial pressure of the cylindrical shells (for $L/R = 5$) with circumferential (n) and axial half (m) wavenumbers. The results for BC-A and BC-D have been presented in Kumar *et al.* [19]. Here, we include them along with the results for BC-B and BC-C for completeness. From Fig. 6, we observe that the static radial pressure obtained analytically are in very good agreement with FEA results obtained using ABAQUS. Qualitative behavior of the variation of critical pressure with wavenumbers (circumferential as well as axial) is the same for all BCs.

Note that, we use a single-mode Galerkin projection to discretize the PDEs. We also check the results using more terms for Galerkin projection and found that the critical pressure does

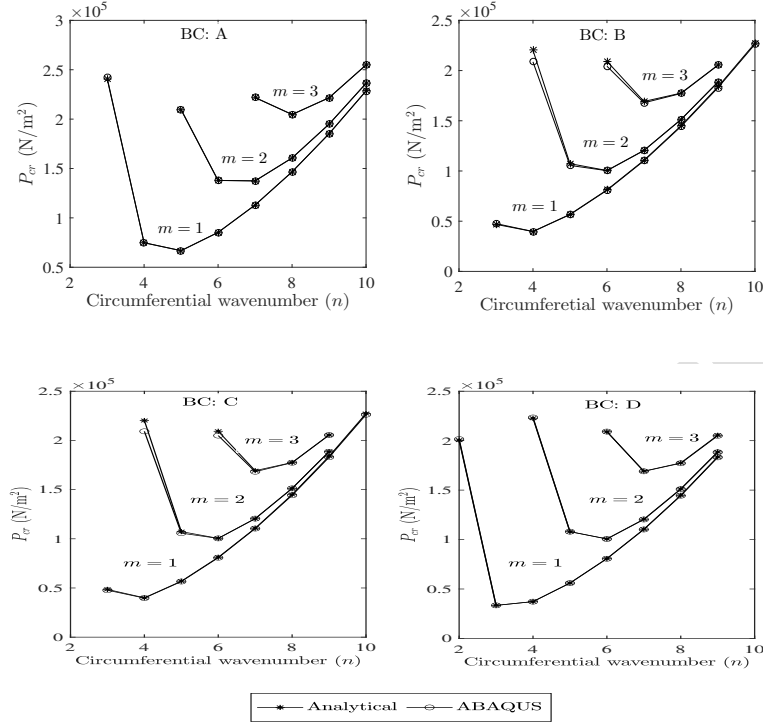


Figure 6: Variations of the critical radial pressure with circumferential and axial half wavenumbers for four BCs, viz., BC-A, BC-B, BC-C and BC-D. Analytical results are compared with FEA results using ABAQUS. Other parameters are $h = 0.01$ m, $R = 2$ m, $L = 10$ m, $E = 2 \times 10^{11}$ N/m², and $\nu = 0.3$.

not change much with inclusion of more terms (details are presented in Appendix A.1). We discuss the convergence study for linear vibration using the various number of terms of the axial mode (presented in Appendix A.2) to justify a single term taken for Galerkin's projection.

3.2 Stability Analysis under Periodic Radial Loading

In this section, we present the stability analysis of the cylindrical shells subjected to time periodic radial pressure ($P = F_0 + F_1 \cos \omega t$) through stability charts in the planes of $F_1 - \omega$ and $F_0 - F_1$, respectively. The geometric and material parameters taken for numerical simulations are $h = 0.01$ m, $R = 2$ m, $L = 2$ m, $E = 2 \times 10^{11}$ N/m², $\rho_m = 7680$ kg/m³, and $\nu = 0.3$ [20]. The buckled shapes corresponding to the lowest critical radial pressure using the mentioned geometric and material parameters are shown in Fig. 7.

The ODEs (25) have time-varying (periodic) coefficients and resemble a higher-dimensional Mathieu-Hill equation [22]. We use Floquet theory to study such system of ODEs and calculate

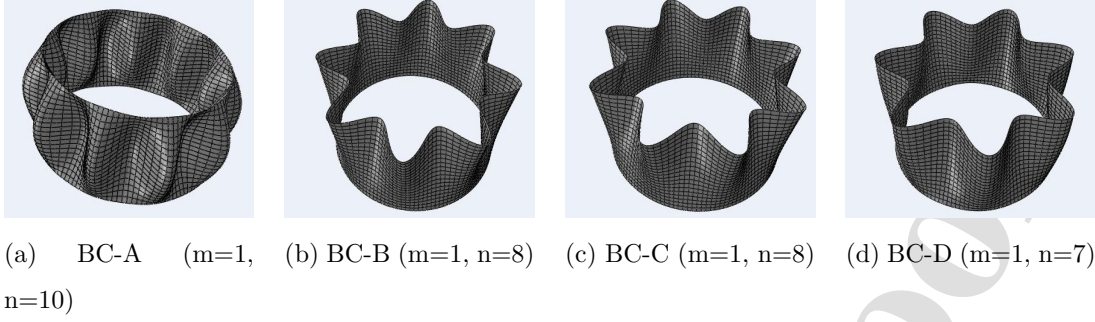


Figure 7: The buckling mode shape corresponding to the lowest critical pressure of cylindrical shells for four different boundary conditions obtained using ABAQUS.

Floquet multipliers to ascertain the stability of the system [22] (discussed in next section) . We present stability charts in the plane of the parameters, viz., the amplitude F_1 and the frequency ω of forcing, and the static part F_0 of the forcing to understand the instability behavior of the cylindrical shells subjected to time varying periodic radial loading. We obtain stability charts for four different boundary conditions mentioned above. We use the same set of geometric and material parameters for all BCs for the stability analysis.

3.2.1 Calculation of Floquet multipliers

In this section, we outline the procedures in brief to calculate the Floquet multipliers [22] to check the instability of the system of ODEs in Eq. (25). We reconsider the system second order ODEs (given in Eq. (25))

$$\begin{bmatrix} \ddot{U}_{mn}(t) \\ \ddot{V}_{mn}(t) \\ \ddot{W}_{mn}(t) \end{bmatrix} = [\mathbb{A}(t)] \begin{bmatrix} U_{mn}(t) \\ V_{mn}(t) \\ W_{mn}(t) \end{bmatrix}, \quad (26)$$

where $[\mathbb{A}(t)]$ is a 3×3 matrix and expressions of its elements are already given in the previous section.

First, we convert the system of second order ODEs to first order ODEs as

$$\dot{\mathbf{X}} = [\mathbb{K}] \mathbf{X}, \quad (27)$$

where $\mathbf{X}$ is a 6×1 vector and given as $\mathbf{X} = [X_1, X_2, X_3, X_4, X_5, X_6]^T$ and X_i are defined in context of Eq. (26) as,

$$X_1(t) = U_{mn}(t), X_2(t) = \dot{U}_{mn}(t), X_3(t) = V_{mn}(t), X_4(t) = \dot{V}_{mn}(t), X_5(t) = W_{mn}(t), X_6(t) = \dot{W}_{mn}(t).$$

The coefficient matrix $[\mathbb{K}]$ is a 6×6 matrix and its elements are,

$$[\mathbb{K}] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \mathbb{A}_{11} & 0 & \mathbb{A}_{12} & 0 & \mathbb{A}_{13} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \mathbb{A}_{21} & 0 & \mathbb{A}_{22} & 0 & \mathbb{A}_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \mathbb{A}_{31} & 0 & \mathbb{A}_{32} & 0 & \mathbb{A}_{33} & 0 \end{bmatrix}, \quad (28)$$

where $\mathbb{A}_{ij} = \mathbb{A}(i, j)$ are elements of the matrix $\mathbb{A}$ and the expressions of $\mathbb{A}(i, j)$ are given in Appendix B.

We assume six sets of initial conditions (ICs) as

$$\begin{aligned} X_1(0) &= [1, 0, 0, 0, 0, 0]^T & X_4(0) &= [0, 0, 0, 1, 0, 0]^T \\ X_2(0) &= [0, 1, 0, 0, 0, 0]^T & X_5(0) &= [0, 0, 0, 0, 1, 0]^T \\ X_3(0) &= [0, 0, 1, 0, 0, 0]^T & X_6(0) &= [0, 0, 0, 0, 0, 1]^T \end{aligned}$$

We solve Eq. (27) for each set of ICs and obtain $X_1(T)$, $X_2(T)$, $X_3(T)$, $X_4(t)$, $X_5(T)$, and $X_6(T)$, those are solutions evaluated at the minimum time period $T = \frac{2\pi}{\omega}$. These column vectors constitute a 6×6 matrix, which is called Floquet matrix $\mathbb{B}$ and given as

$$\mathbb{B} = [X_1(T), X_2(T), X_3(T), X_4(t), X_5(T), X_6(T)] . \quad (29)$$

The eigenvalues of the Floquet matrix $\mathbb{B}$ are Floquet multipliers which predict the stability of the system of ODEs in Eq. (26). The system is stable when all Floquet multipliers are less than unity in magnitude and unstable if one or more Floquet multipliers are greater than unity in magnitude. We solve Eq. (27) numerically using MATLAB solver ode45 to obtain Floquet multipliers.

3.2.2 Stability charts for simply supported boundary condition (BC-A)

For BC-A, we find the mode corresponding to $(m, n) = (1, 10)$ is the first buckling mode. However, the wavenumber corresponding to the lowest natural frequency is given by $(m, n) = (1, 8)$ [20]. Thus to get a complete picture of the instability of cylindrical shells, we present the stability charts for four different circumferential wavenumbers ($n = 8, 9, 10, 11$). Here the stable and unstable regions are represented by blue and red colors, respectively.

Figures 8a-8h show the stability charts in plane $F_1 - \omega$ with different values of F_0 and circumferential wavenumber. In the left column of Fig. 8, all the panels show stability charts with zero static part of the radial pressure (i.e., $F_0 = 0$), and the right columns represent the stability charts with a static radial pressure $F_0 = 1.75 \times 10^5$ N/m², which is half of the lowest critical pressure. The stability charts for finite cylinders with simply supported ends are qualitatively similar to that for infinitely long circular cylindrical shells. We identify large unstable regions occur around the forcing frequency which is equal to the twice of the natural frequency corresponding to each circumferential wavenumber. When static radial pressure is inward, the unstable regions shift towards the left, that is towards the lower frequency range. This shift is consistent with observations made by Kumar *et al.* [20], where a radially inward pressure reduces the natural frequency corresponding to each circumferential wavenumber. The frequency bandwidth for the unstable regime for a fixed forcing amplitude F_1 increases with an increase in F_0 , manifested by the widening of the stability regimes between the left and right columns. Furthermore, the stability boundaries are curved in higher F_1 range when F_0 is present (the stability boundaries appear straight when $F_0 = 0$). Figure 8a shows that the principal parametric resonance occurs at the forcing frequency $\omega = 2\omega_8 = 876$ ($\omega_8 = 438$ rad/sec is the natural frequency corresponding to $n = 8$ and $m = 1$). The instability tongues emerge at forcing frequencies $\omega = 876, 438, 292, 219, 175.2$ and so on as shown in Fig. 8a. Therefore, the parametric resonance occurs when the forcing frequency ω assumes the form $\omega = \frac{2\omega_n}{k}$, where ω_n is the fundamental frequency of the cylinder corresponding to the circumferential wavenumber n and k is a positive integer. We observe the similar trends in the stability charts for other circumferential and axial wavenumbers in the plane $F_1 - \omega$.

For finite cylindrical shells, the first few natural frequencies are close to each other and hence, can be assumed to belong to a cluster. Also, the circumferential wavenumbers corresponding to the lowest natural frequency and lowest critical radial pressures are different [20]. Accordingly, to get a better idea of the stable and the unstable regions near circumferential wavenumbers corresponding to the lowest natural frequency and lowest critical radial pressure, we combine the stability charts given in Figs. 8a-8h into a single stability chart for two different values of F_0 ($= 0$ and 1.75×10^5 N/m²) as shown in Figs. 9a and 9b. The combined chart marks the stable and the unstable regions for four consecutive circumferential wavenumbers $n = 8, 9, 10, 11$, and include those corresponding to the lowest natural frequency ($n = 8$)

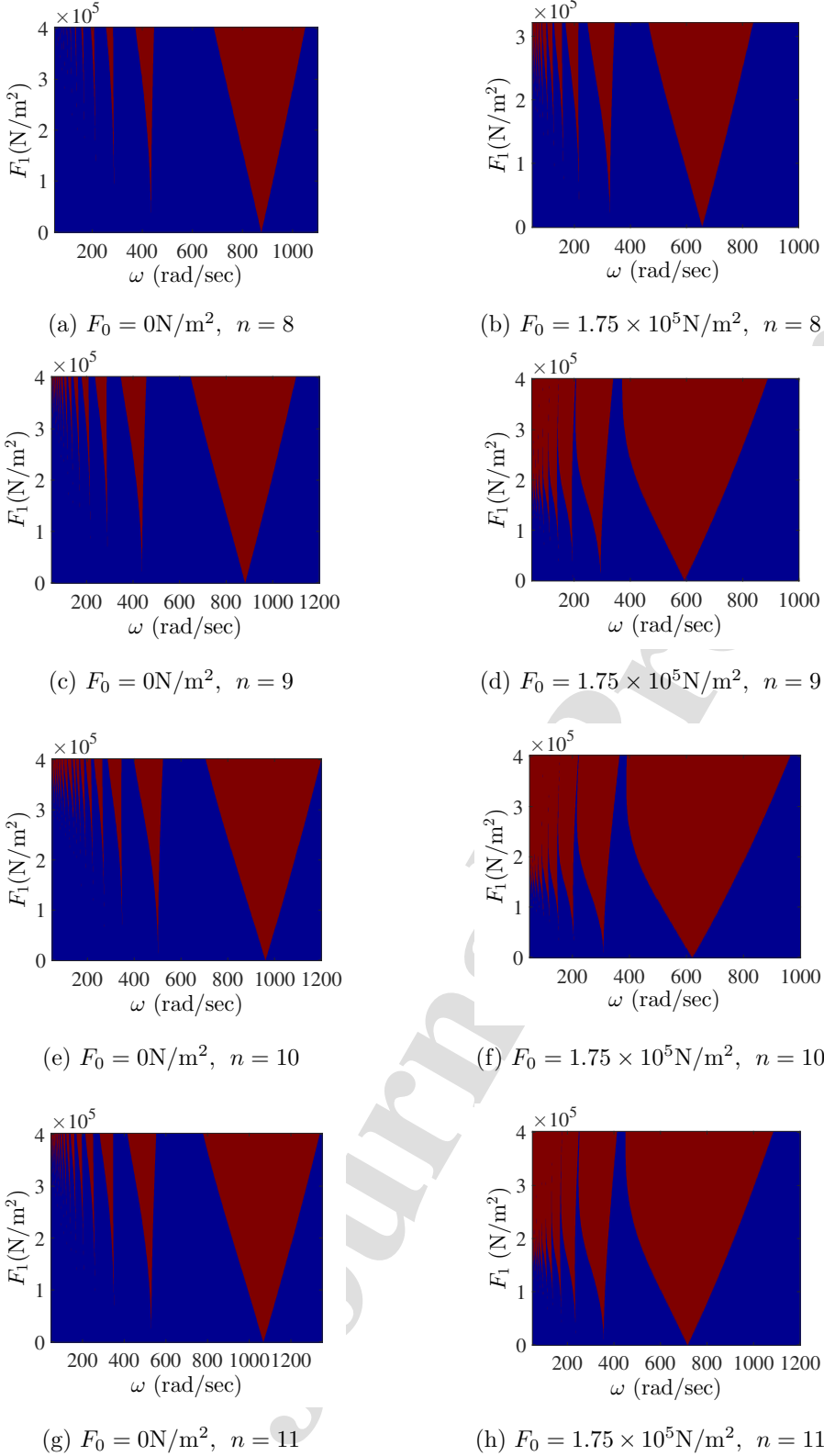
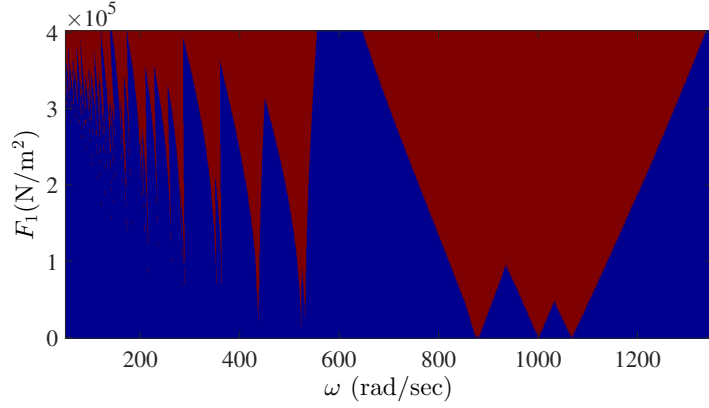
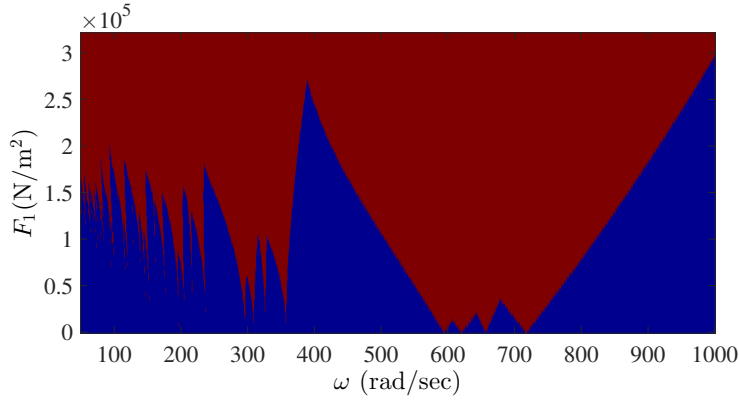


Figure 8: Stability charts for the circular cylindrical shell with simply supported ends (BC-A) in the plane F_1 - ω at $F_0 = 0 \text{ N/m}^2$ and $F_0 = 1.75 \times 10^5 \text{ N/m}^2$ for circumferential wavenumbers $n = 8, 9, 10, 11$ and axial half wavenumber $m = 1$.

and lowest critical radial pressure ($n = 10$). The combined stability charts (Figs. 9a and 9b) exhibit a much finer structure for stable and unstable regimes. From Figs. 9a and 9b, we conclude that there are few large frequency bandwidths corresponding to the parametric instability of a shell for higher F_1 whereas, for smaller F_1 , there are several smaller frequency bandwidths in which the axisymmetric solution becomes unstable.



(a) $F_0 = 0 \text{ N/m}^2$, $n = 8, 9, 10, 11$



(b) $F_0 = 1.75 \times 10^5 \text{ N/m}^2$, $n = 8, 9, 10, 11$

Figure 9: Combined stability charts for the circular cylindrical shell with simply supported ends (BC-A) in the plane $F_1 - \omega$ at $F_0 = 0 \text{ N/m}^2$ and $F_0 = 1.75 \times 10^5 \text{ N/m}^2$ for circumferential wavenumbers $n = 8, 9, 10, 11$ and axial half wavenumber $m = 1$.

To verify the stability charts, we present the time history of the radial deflection for different forcing amplitude and excitation frequencies in stability corresponding to $n = 8$ and $F_0 = 0$ (corresponding to Fig. 8a) and $1.75 \times 10^5 \text{ N/m}^2$ (corresponding to Fig. 8b) as shown in Figs. 10 and 11, respectively .

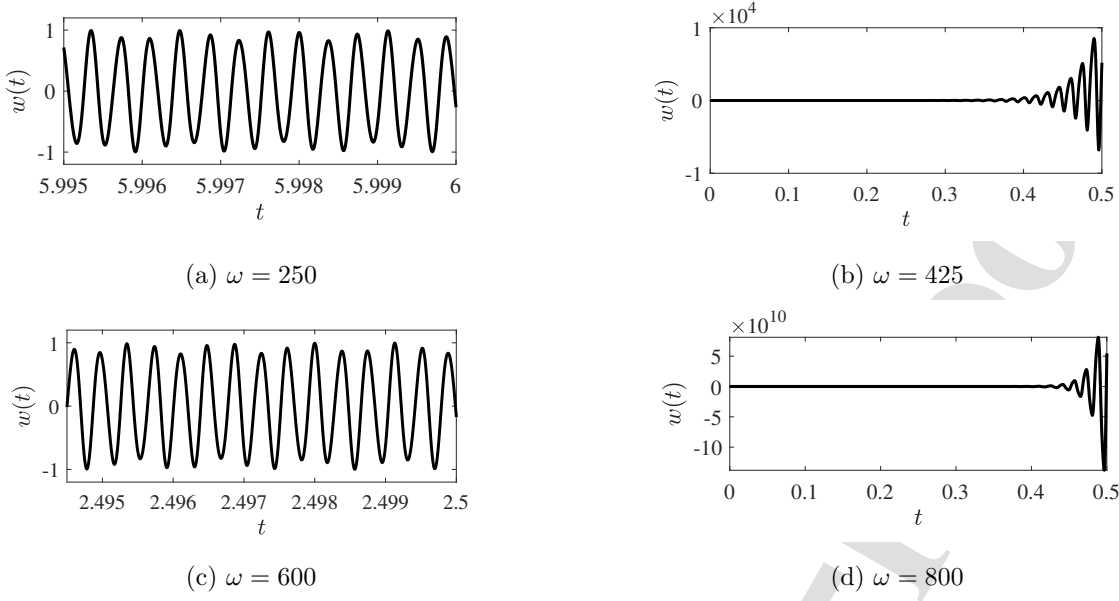


Figure 10: Time response of the system for different forcing frequency ω and for $F_1 = 2 \times 10^5 \text{ N/m}^2$, $F_0 = 0$ and $n=8$.

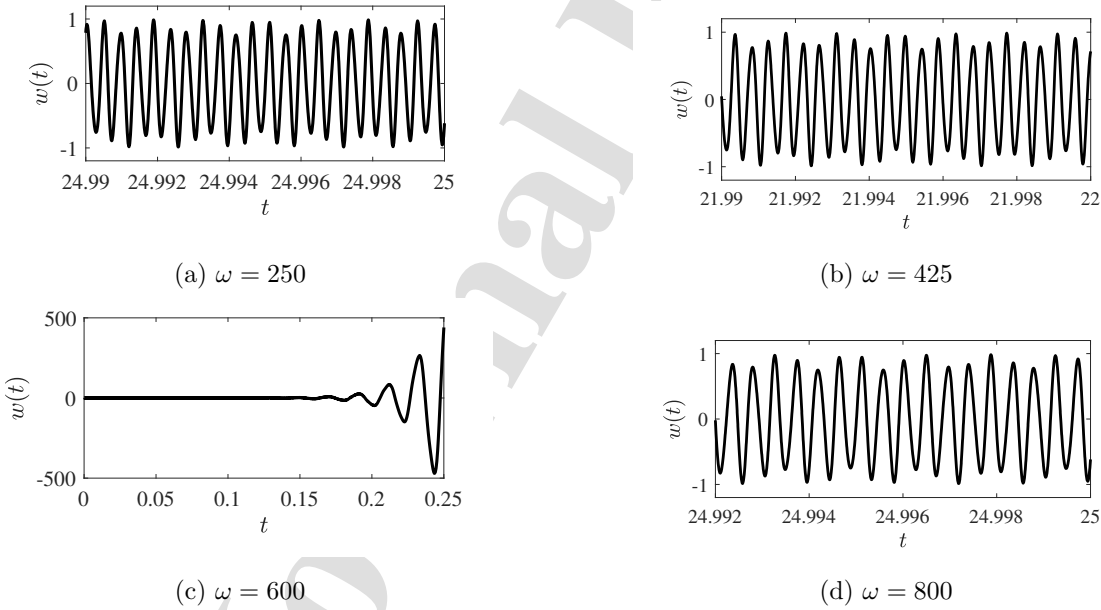


Figure 11: Time response of the system for different forcing frequency ω and for $F_1 = 2 \times 10^5 \text{ N/m}^2$, $F_0 = 1.75 \times 10^5 \text{ N/m}^2$.

Figure 10 presents time history of the present model corresponding to wavenumbers $m = 1$, $n = 8$ (in contrast to Fig. 8a). We choose a fixed value of $F_0 = 0$ and $F_1 = 2 \times 10^5 \text{ N/m}^2$ and vary the excitation frequency $\omega = 250, 425, 600, 800 \text{ rad/sec}$ and plot time histories

as shown in Figs. 10a-10d. The behavior of these time histories are consistent (in terms of bounded and unbounded response) with the stability chart presented in Fig. 8a for $n = 8$. Similarly, we choose different set of F_0 and F_1 ($F_0 = 1.75 \times 10^5$ N/m² and $F_1 = 2 \times 10^5$ N/m²) and present time history for the same set of the excitation frequencies as shown in Figs. 11a-11d. Again we find the time history presented in Fig. 11 are consistent with the stability chart presented in Fig. 8b.

Figure 12a-12d show stability charts in the plane F_0-F_1 at a fixed value of forcing frequency ($\omega = 440$ rad/sec) for four consecutive circumferential wavenumbers, viz., $n = 8, 9, 10, 11$. From Figs. 12a-12d, we can easily find the radial pressure corresponding to static buckling for the various circumferential wavenumbers $n = 8, 9, 10, 11$. In each of these figures, the unstable regions to the extreme right starting from the point $F_0 = p_{cr}$ and $F_1 = 0$ corresponds to the case of static buckling under radial pressure. These regimes are clearly visible in the cases of $n = 9, 10$ and 11 while it is barely evident in the case of $n = 8$. It is also clear from Figs. 12a-12d that the static critical radial pressure is minimum for the circumferential wavenumber $n = 10$. We present the time series of various forcing parameters corresponding to the Fig. 12a in Figs. 13a-13d to justify stable and unstable regions.

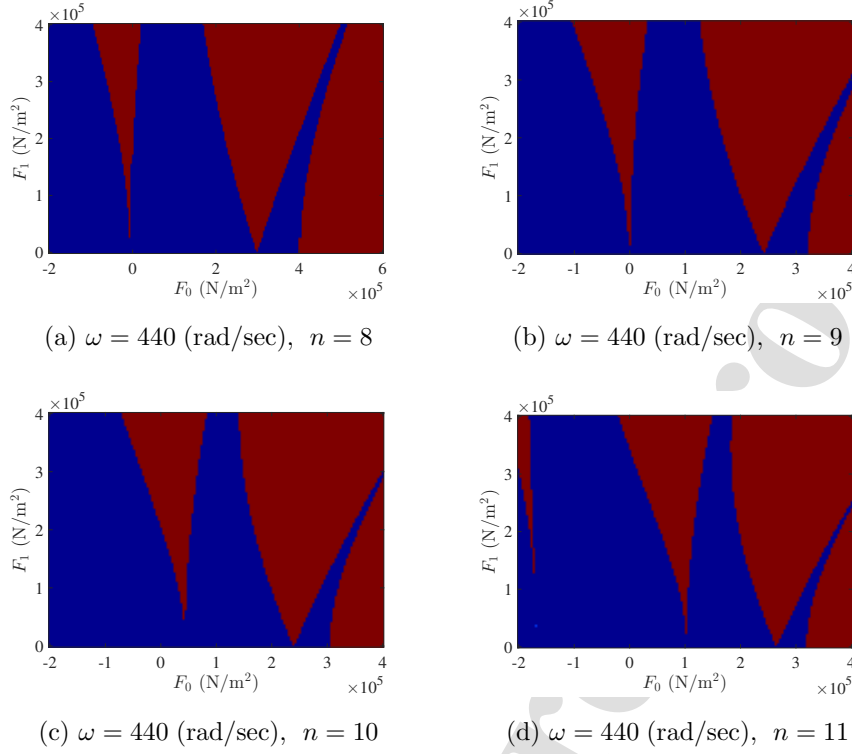


Figure 12: Stability charts for the circular cylindrical shell with simply supported ends (BC-A) in the plane $F_0 - F_1$ for $\omega = 440$ rad/sec, $n = 8, 9, 10, 11$ and $m = 1$.

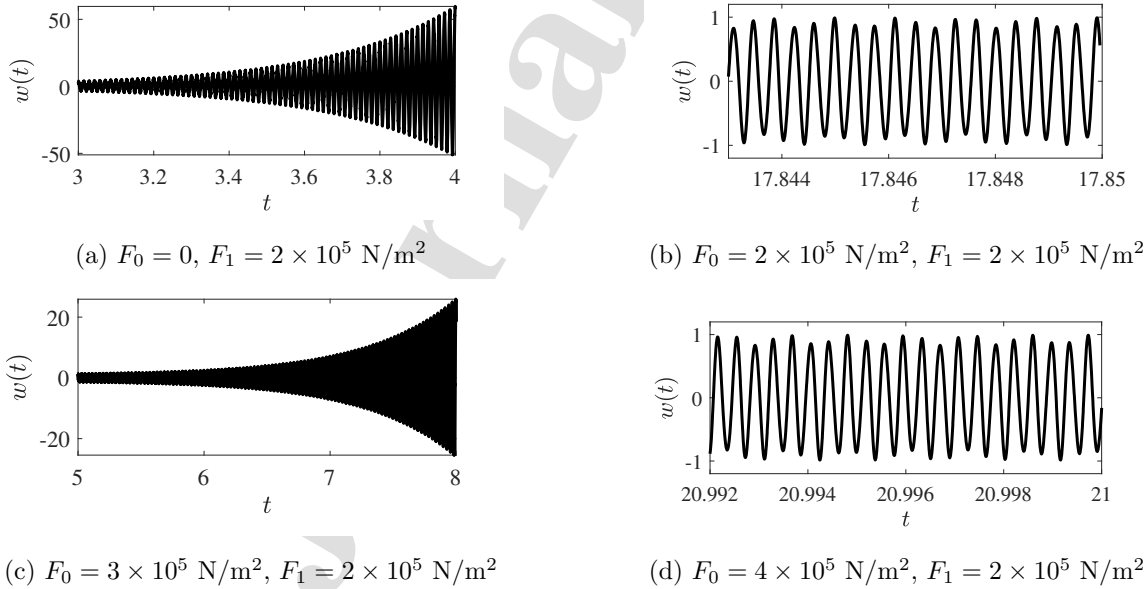


Figure 13: Time response of the system for different forcing parameters for $n = 8$ and $\omega = 440$ (see Fig. 12a).

Again, to get a better idea of the stable and unstable regions for circumferential wavenumbers corresponding to the lowest natural frequency and lowest critical radial pressure in $F_0 - F_1$ plane, we combine stability chart for four consecutive circumferential wavenumbers (as shown in Figs. 12a-12d) in Fig. 14. It can be observed from the combined stability chart, shown in Fig. 14, that many instability tongues have appeared before the onset of static buckling. We observe a significant effect on the effective minimum static buckling pressure, that is, a decrease in the effective minimum static buckling load to around 2.5×10^5 N/m² from 3.5×10^5 N/m² for the critical circumferential wavenumber of $n = 10$. Another interesting observation from this combined stability chart is that because of the overlapping instability tongues, the maximum F_1 value required to initiate parametric instability decreases even if the value of the static component of loading F_0 is small. There is an instability region starting from very low F_1 values for $F_0 = 0$ which actually corresponds to the circumferential wavenumber of $n = 9$ which neither corresponds to the lowest natural frequency nor the static buckling load. Thus, the circumferential wavenumber of $n = 9$ becomes the first critical wavenumber for purely dynamic loading. Thus we can conclude that the circumferential wavenumber corresponding to the first point of instability can be different from the circumferential wavenumber corresponding to the lowest critical pressure or lowest natural frequency.

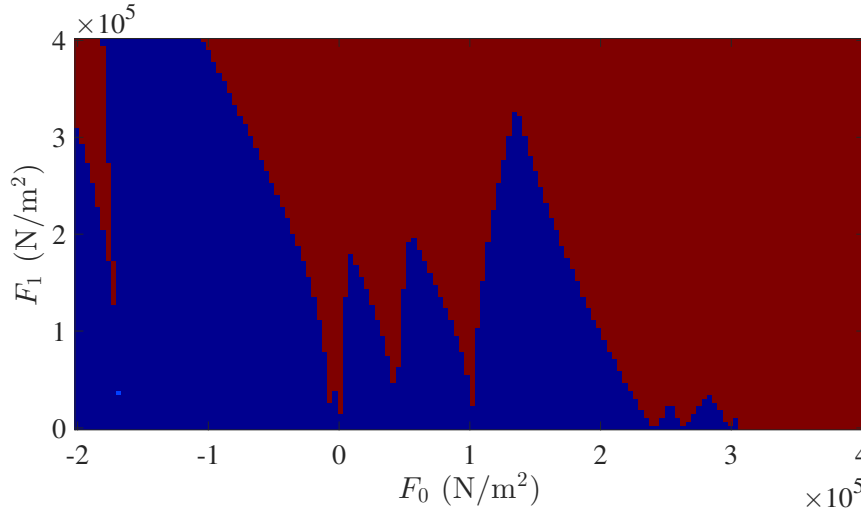


Figure 14: Combined stability charts for the circular cylindrical shell with simply supported ends (BC-A) in the plane $F_0 - F_1$ at $\omega = 440$ rad/sec for $n = 8, 9, 10, 11$ and $m = 1$.

3.2.3 Stability charts for BC-A using Yao's model [9]

In this section, we present stability charts of the cylindrical shell with simply supported boundary condition using Yao's model [9] and compare with the present model. Towards this, first we present the variation of the natural frequencies and associated mode shapes with radial static pressure and compare with FEA results (using ABAQUS) and results presented by Kumar et al [20].

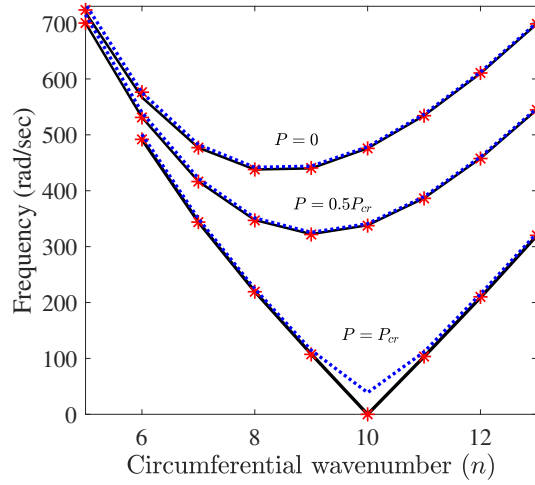


Figure 15: Variation of the natural frequencies and associated mode shape with radial pressure. Results obtained using present model [20] and using Yao's model [9]. Black solid lines (—) for present model, blue dotted lines ($\cdots$) for Yao's model [9] and red asterisk (*) for FEA results.

Figure 15 shows the variation of the natural frequencies and associated mode shape with static radial pressure which are obtained using present model and Yao's model [9], respectively. Results show the present model is in very good agreement with FEA results (using ABAQUS) which has already been shown by the first author in his previous work (Kumar et al [20]). From Fig. 15, it is clear that Yao's model (with zero static radial pressure) is also in very good agreement with FEA results but as we increase the static radial pressure, we observe disagreement with FEA as well as with present model. Specially, when static radial pressure reaches to the critical radial pressure the lowest natural frequency obtained using FEA and present model becomes zero but Yao's model [9] gives 38 rad/sec. This disagreement occurs due neglecting pre-stress radial deflection and assuming constant hoop stress (i.e., independent of special coordinates). Thus we conclude that the present model is more reliable for the

cylindrical shells subjected to static radial pressure.

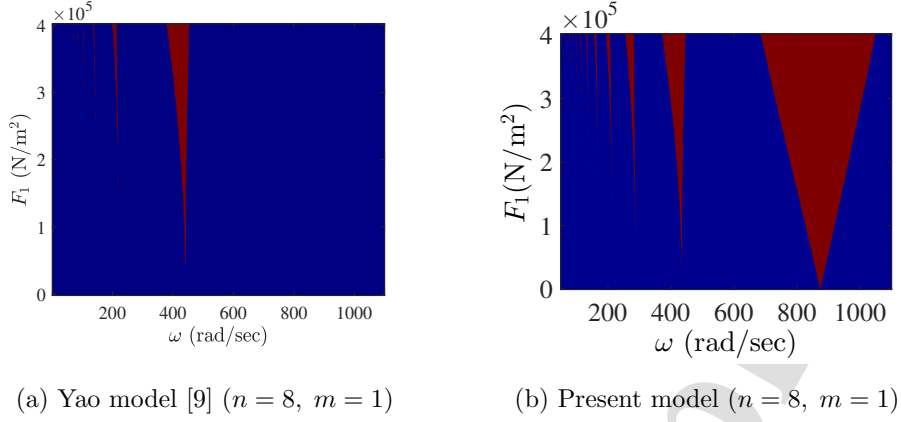


Figure 16: Stability charts obtained using Yao's model [9] in the plane $F_1 - \omega$ and $F_0 = 0 \text{ N/m}^2$.

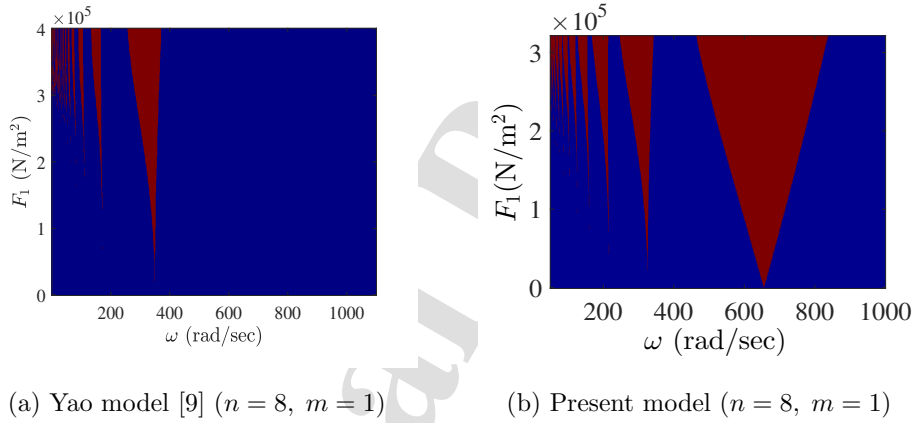


Figure 17: Stability charts obtained using Yao's model [9] in the plane $F_1 - \omega$ and $F_0 = 1.75 \times 10^5 \text{ N/m}^2$.

Next, we present stability charts for simply-supported cylindrical shells subjected to periodic radial pressure ($P = F_0 + F_1 \cos \omega t$). Figures 16a and 16b present stability charts in the plane of $F_1 - \omega$ at $F_0 = 0$ obtained using Yao's model [9] and the present model. Here we observe that Yao's model fails to predict the instability tongue near principle parametric resonance i.e., when the excitation frequency becomes equal to twice of the natural frequency of the system, which is clearly visible in Figs. 16. Yao's model predicts all instability tongues properly except near the excitation frequency equal to twice of the natural frequency although system wider instability region near $\omega = 2\omega_n$. Similar behavior has been observe when $F_0 = 1.75 \times 10^5 \text{ N/m}^2$ as shown in Fig. 17.

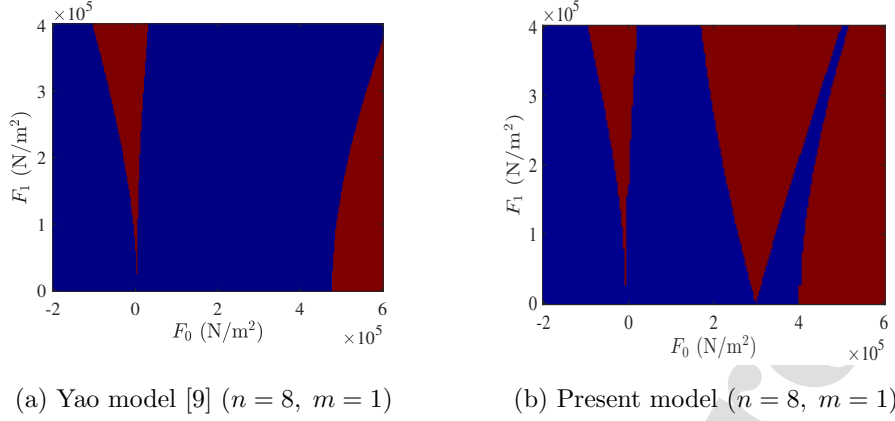
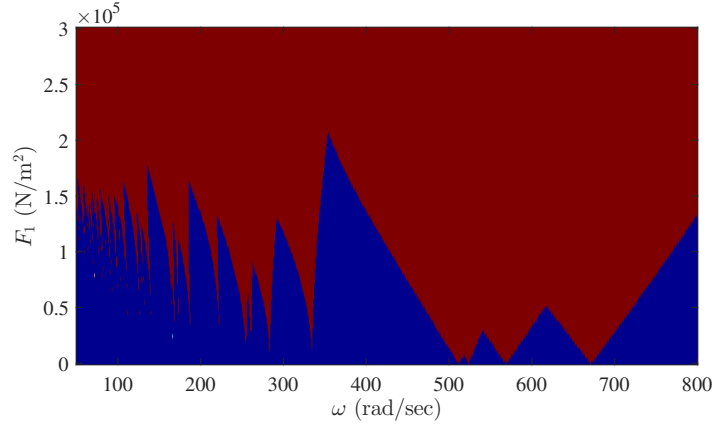


Figure 18: Stability charts obtained using Yao's model [9] in the plane $F_1 - F_0$ and $\omega = 440$ rad/sec.

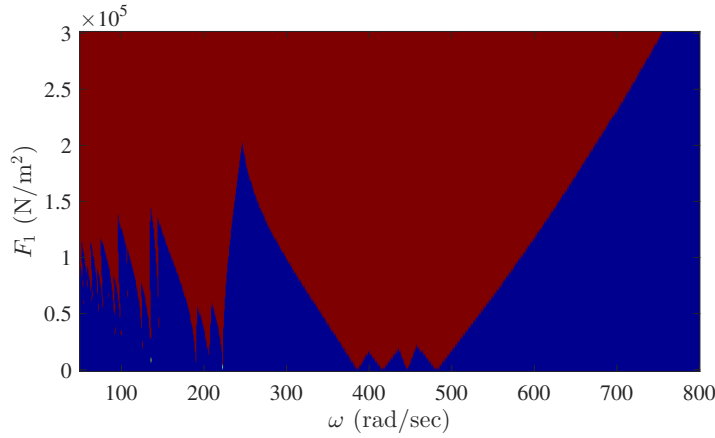
Figures 18a and 18b, present stability charts in the plane $F_0 - F_1$ and $\omega = 440$ rad/sec. Again we observe Yao's model [9] predict instability tongue at $F_0 = 0$, which is near primary resonance (i.e. $\omega = \omega_n$). As we increase the static component (F_0), the natural frequency associated with $n = 8$ decreases and the current excitation frequency $\omega = 440$ rad/sec reached near twice of the natural frequency again unstable region initiate which is clearly shown in Fig. 18b, but Yao's model [9] fails to predict such instability region. Through the stability charts presented in this section, we conclude that the assumptions made by Yao [9] is not sufficient to predict all instability tongues and present model shows consistent behaviors.

3.2.4 Stability charts for BC-B, BC-C and BC-D boundary conditions

In this section, we present stability analysis of the cylindrical shells with boundary conditions BC-B, BC-C and BC-D, respectively. For BC-B, the circumferential wavenumbers corresponding to the lowest fundamental frequency ($\omega = 259.21$) and the lowest critical radial pressure ($P_{cr} = 2.03548 \times 10^5$ N/m²) are $n = 7$ and $n = 8$, respectively. The natural frequencies corresponding to the circumference wavenumbers $n = 6, 7, 8, 9$ are closely spaced for both cases (e.g, $F_0 = 0$ and $F_0 = 10^5$ N/m²) (see Kumar et al [20]). The combined stability charts for $n = 6, 7, 8, 9$ for $F_0 = 0$ and $F_0 = 10^5$ N/m², depicting the complete dynamic behavior, have been shown in Figs. 19a and 19b.



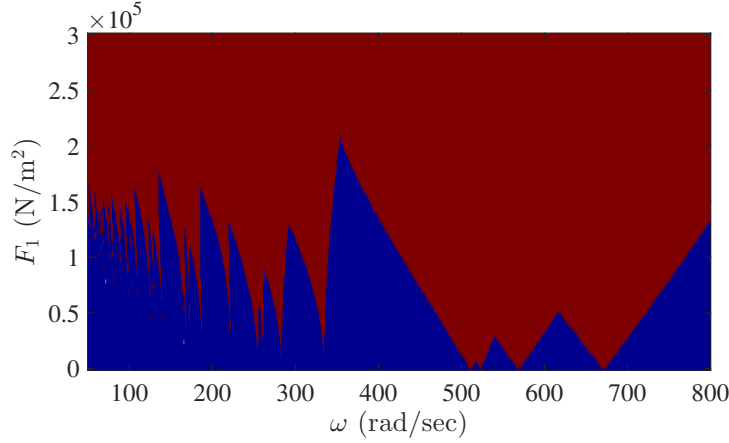
(a) $F_0 = 0 \text{ N/m}^2$, $n = 6, 7, 8, 9$



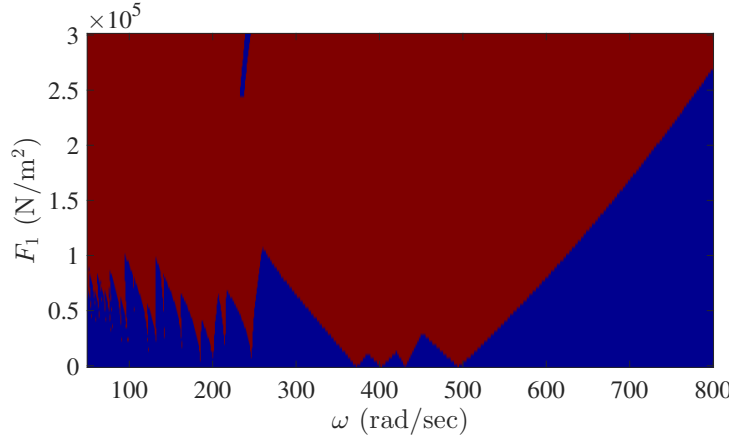
(b) $F_0 = 10^5 \text{ N/m}^2$, $n = 6, 7, 8, 9$

Figure 19: Combined stability charts for the circular cylindrical shell with pinned-free ends with axial constraint at the pinned end (BC-B) in the plane $F_1 - \omega$ at $F_0 = 0 \text{ N/m}^2$ and $F_0 = 10^5 \text{ N/m}^2$ for different circumferential wavenumbers $n = 6, 7, 8, 9$ and axial half wavenumber $m = 1$.

Figures 20a and 20b show the combined stability charts for BC-C for four consecutive circumferential wavenumber $n = 6, 7, 8, 9$ in the plane of $F_1 - \omega$ at two fixed values of the static component of the forcing, i.e., $F_0 = 0$ and $F_0 = 10^5 \text{ N/m}^2$ (which is half of the lowest critical radial pressure of BC-C). The group of these four circumferential wavenumbers includes the circumferential wavenumbers corresponding to the lowest natural frequency ($n = 7$) and the lowest critical pressure ($n = 8$).



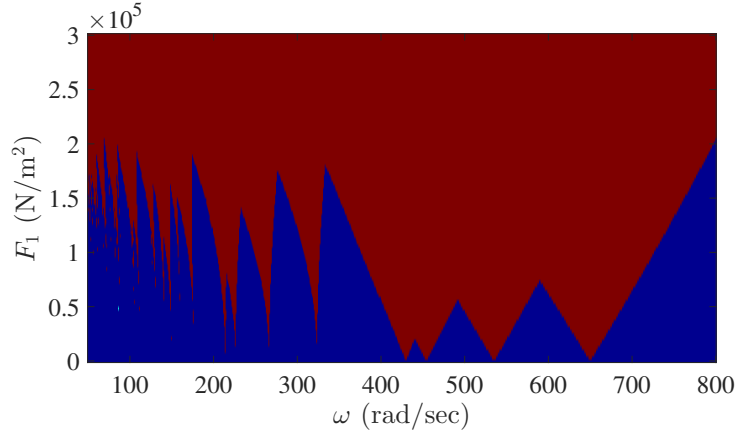
(a) $F_0 = 0 \text{ N/m}^2$, $n = 6, 7, 8, 9$



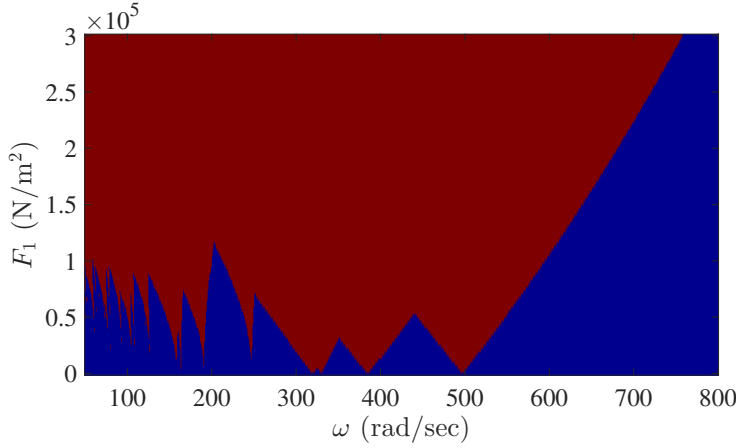
(b) $F_0 = 10^5 \text{ N/m}^2$, $n = 6, 7, 8, 9$

Figure 20: Combined stability charts for the circular cylindrical shell with fixed-free ends (BC-C) in the plane $F_1 - \omega$ at $F_0 = 0 \text{ N/m}^2$ and $F_0 = 10^5 \text{ N/m}^2$ for different circumferential wavenumbers $n = 6, 7, 8, 9$ and axial half wavenumber $m = 1$.

The lowest natural frequency of the cylindrical shell with fixed free (BC-C) boundary condition is $\omega = 259 \text{ rad/sec}$ and corresponding circumferential wavenumber is $n = 7$ and the lowest critical radial pressure is $2.03 \times 10^5 \text{ N/m}^2$ and associated circumferential wavenumber is $n = 8$. The natural frequency corresponding to $n = 8$ is $\omega = 286.6 \text{ rad/sec}$.



(a) $F_0 = 0 \text{ N/m}^2$, $n = 6, 7, 8, 9$



(b) $F_0 = 10^5 \text{ N/m}^2$, $n = 6, 7, 8, 9$

Figure 21: Combined stability charts for the circular cylindrical shell with pinned-free with axial constraint at the free end (BC-D) in the plane $F_1 - \omega$ at $F_0 = 0 \text{ N/m}^2$ and $F_0 = 0.75 \times 10^5 \text{ N/m}^2$ for different circumferential wavenumbers $n = 6, 7, 8, 9$ and axial half wavenumber $m = 1$.

Figures 21a and 21b show the combined stability charts for BC-D in the plane $F_1 - \omega$ for four consecutive circumferential wavenumbers $n = 6, 7, 8, 9$ which includes circumferential wavenumbers corresponding to the lowest natural frequency ($n = 6$) and the lowest critical radial pressure ($n = 7$) for BC-D. We present, the stability charts in the plane $F_1 - \omega$ for two different values of the static component of the excitation forcing, e.g., $F_0 = 0$ and $F_0 = 0.75 \times 10^5 \text{ N/m}^2$ (which is the half of the lowest critical pressure of BC-D). We observe from Fig. 21a, the stability of the cylindrical shells is very sensitive to the excitation when the

excitation frequency lies between $\omega = 400$ to 700 rad/sec. These frequency ranges become wider as we increase the static component of the forcing (F_0), e.g., we find a wider sensitive frequency range for $F_0 = 0.75 \times 10^5$ N/m² (as shown in Fig. 21b) as compared to when there was zero static components ($F_0 = 0$) as shown in Fig. 21a. The stability charts in the plane of $F_0 - F_1$ with fixed values of excitation frequencies (lowest natural frequency for the respective BCs) have also been presented (see Appendix B) to understand the dynamic behavior in $F_0 - F_1$ plane too. Through the stability charts in plane $F_0 - F_1$, we can easily find the condition of static buckling ($F_1 = 0$ and $F_0 = P_{cr}$) for each circumferential wavenumber for the respective BCs, e.g., for BC-B and C ($n = 8$, $F_0 = 2 \times 10^5$) and BC-D ($n = 7$, $F_0 = 1.5 \times 10^5$). Along with combined stability charts in the plane $F_0 - F_1$ (see Appendix B), we also present stability charts in the plane of $\omega - F_1$ with fixed values of F_0 for individual circumferential wavenumber in Appendix B. Thus we present a detail stability analysis of the cylindrical shells subjected to time varying radial pressure, which would be helpful during design of the cylindrical shells under such loading conditions.

4 Conclusion

In this paper, we discuss the parametric instability of circular cylindrical shells subjected to time-varying (periodic in time) radial pressure for four different boundary conditions viz., simply supported, pinned-free (axial constraint at the pinned end), fixed-free, and pinned-free (axial constraint at the free end). First, we present the variations of static buckling pressure (uniform) with wavenumbers (m and n) of thin-walled circular cylindrical shells for different boundary conditions (as mentioned above). The results for the critical radial pressure obtained analytically are verified by FEA results obtained using ABAQUS and they are in a very good agreement. Next, we study the dynamic instabilities of the cylindrical shells subjected to the periodic radial pressure (having static and fluctuating parts). We obtain the results for the dynamic instability of the cylindrical shells through stability charts in the plane of different forcing parameters, (namely, the static component of the pressure, the amplitude, and frequency of the fluctuating component of the pressure) where we show sets of the forcing parameters for which system has bounded (stable) and unbounded (unstable) responses. From stability charts for the individual circumferential wavenumber, we observe that the excitation frequency corresponding to the initiation of each instability tongue are closely packed for all

wavenumbers. Thus, to get a broader understanding of the instability, we present a combined stability chart for each plane and BCs. Previous studies by Yao [9] and Lindberg and Florance [11] have linearized the nonlinear PDEs with constant axial stress and hoop stress, which can be an approximation for $L/R > 4$, but for short cylinder such approximation is not advisable. The present work linearize the nonlinear dynamic equation along axial, circumferential and radial directions with steady state displacement's components which is more consistent and application for any range of L/R .

We also observe a trend for the initiation of the instability tongues with forcing frequency in the stability chart in the plane $F_0 - \omega$. We find, the response of the system becomes unbounded when the forcing frequency satisfies a unique relation with natural frequency, i.e., $\omega = \frac{2\omega_{mn}}{k}$. It has been observed that the wider unstable regions are found when forcing frequency reaches twice the natural frequency of the cylindrical shells, i.e., for $k = 1$ and the instability tongues become narrow as we move towards the lower excitation frequency, i.e. for larger values of k . Such observations are consistent with the observation from the simple Mathieu-Hill equation. Unstable regions are shifted towards lower forcing frequency range as we increase the values of the static component (F_0), which is also consistent with results shown by Kumar *et al.* [20]. It has been also observed that the incorporation of a static load increases the frequency bandwidth corresponding to the parametric instability for a fixed magnitude of the variable load. We observe same qualitative behavior for all BCs analyzed in the present work. Thus, these stability charts and the observations from them will be very useful in the design of circular cylindrical shell components depending on the information on the dynamic loading situations.

Appendix A: Convergence study with numbers of mode used to discretize linear PDEs.

A.1: Static buckling pressure

We demonstrate the convergence with number of axial modes given below in Eq. (30),

$$\tilde{u} = \sum_{m=1}^M U_{mn}(t) \cos\left(\frac{\beta_m}{L}x\right) \cos(n\theta), \quad (30a)$$

$$\tilde{v} = \sum_{m=1}^M V_{mn}(t) \sin\left(\frac{\beta_m}{L}x\right) \sin(n\theta), \quad (30b)$$

$$\tilde{w} = \sum_{m=1}^M W_{mn}(t) \sin\left(\frac{\beta_m}{L}x\right) \cos(n\theta), \quad (30c)$$

where M denotes number of axial modes taken during Galerkin projection. We tabulate the critical pressure for four consecutive circumferential wavenumbers (those are corresponding to the four lowest critical pressure of the respective BCs) and axial half wavenumber $m = 1$ in Tables 1 through 3. These results are obtained using different number of axial modes taken during Galerkin projection (i.e., different values of M in Eq. (30)). We do not find any variations in the numerical values of the critical radial pressure with respect to the number of axial modes chosen during Galerkin projection as shown in Tables 1 through 3. The present convergence analysis allows us to use single term of the axial mode during the Galerkin projection for onward analysis in the present paper without affecting the results (qualitatively and quantitatively, both).

Table 1: Convergence analysis for the critical pressure with respect to number of axial modes taken for Galerkin projection (for BC-A). Here $E = 2 \times 10^{11}$ N/m², $\rho_m = 7680$ Kg/m³, $\nu = 0.3$, $L = 2$ m, $R = 2$ m, and $h = 0.01$ m.

Number of axial modes (M)	P_{cr} (N/m ²), n=7	P_{cr} (N/m ²), n=8	P_{cr} (N/m ²), n=9	P_{cr} (N/m ²), n=10
1	7.37933×10^5	4.72646×10^5	3.76166×10^5	3.53643×10^5
2	7.37933×10^5	4.72646×10^5	3.76166×10^5	3.53643×10^5
3	7.37933×10^5	4.72646×10^5	3.76166×10^5	3.53643×10^5
4	7.37933×10^5	4.72646×10^5	3.76166×10^5	3.53643×10^5
5	7.37933×10^5	4.72646×10^5	3.76166×10^5	3.53643×10^5

Table 2: Convergence analysis for the critical pressure with respect to number of axial modes taken for Galerkin projection (for BC-B). Here $E = 2 \times 10^{11}$ N/m², $\rho_m = 7680$ Kg/m³, $\nu = 0.3$, $L = 2$ m, $R = 2$ m, and $h = 0.01$ m.

Number of axial modes (M)	P_{cr} (N/m ²), n=6	P_{cr} (N/m ²), n=7	P_{cr} (N/m ²), n=8	P_{cr} (N/m ²), n=9
1	3.22694×10^5	2.19616×10^5	2.03548×10^5	2.20987×10^5
2	3.22762×10^5	2.19664×10^5	2.03573×10^5	2.21070×10^5
3	3.22728×10^5	2.19633×10^5	2.03561×10^5	2.20991×10^5
4	3.22750×10^5	2.19642×10^5	2.03566×10^5	2.21003×10^5
5	3.22750×10^5	2.19642×10^5	2.03566×10^5	2.21003×10^5

Table 3: Convergence analysis for the critical pressure with respect to number of axial modes taken for Galerkin projection (for BC-D). Here $E = 2 \times 10^{11}$ N/m², $\rho_m = 7680$ Kg/m³, $\nu = 0.3$, $L = 2$ m, $R = 2$ m, and $h = 0.01$ m.

Number of axial modes (M)	P_{cr} (N/m ²), n=5	P_{cr} (N/m ²), n=6	P_{cr} (N/m ²), n=7	P_{cr} (N/m ²), n=8
1	3.77025×10^5	2.07719×10^5	1.69061×10^5	1.77394×10^5
2	3.55934×10^5	1.86539×10^5	1.51709×10^5	1.59694×10^5
3	3.55934×10^5	1.86539×10^5	1.51709×10^5	1.59694×10^5
4	3.55934×10^5	1.86539×10^5	1.51709×10^5	1.59694×10^5
5	3.55934×10^5	1.86539×10^5	1.51709×10^5	1.59694×10^5

A.2: Natural frequency

In this section, we compute the natural frequencies after taking different number of axial modes during Galerkin projection. Figures 22 through 24 show the convergence analyses of natural frequencies for BC-A, BC-B, BC-C and BC-D with respect to number of axial modes. It has also been noticed in Figs. 22 through 24, the variation in the numerical values of natural frequencies obtained by considering a different number of modes for axial displacement M is an order of 10^{-5} . Thus we use single term of the axial mode (i.e., $M = 1$) at a time during Galerkin projection to avoid tedious calculations and reduce the computational time.

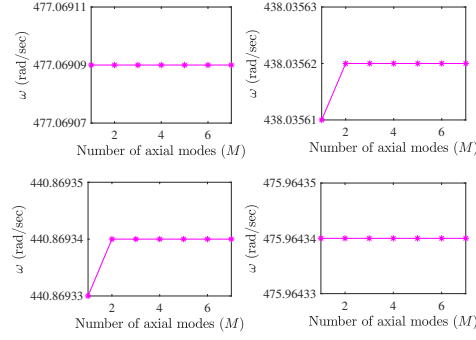


Figure 22: Convergence of natural frequencies with respect to number of axial modes taken during Galerkin projection for BC-A. Here $E = 2 \times 10^{11}$ N/m², $\rho_m = 7680$ kg/m³, $\nu = 0.3$, $L = 2$ m, $R = 2$ m, and $h = 0.01$ m.

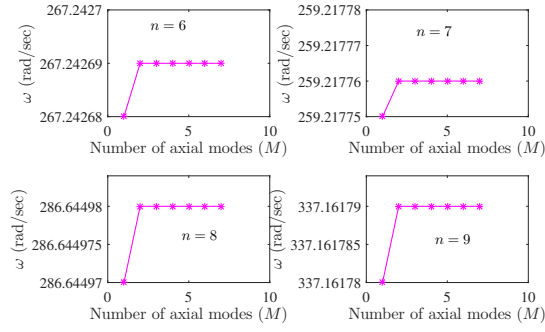


Figure 23: Convergence of natural frequencies with respect to number of axial modes taken during Galerkin projectio (BC-B and BC-C). Here $E = 2 \times 10^{11}$ N/m², $\rho_m = 7680$ kg/m³, $\nu = 0.3$, $L = 2$ m, $R = 2$ m, and $h = 0.01$ m.

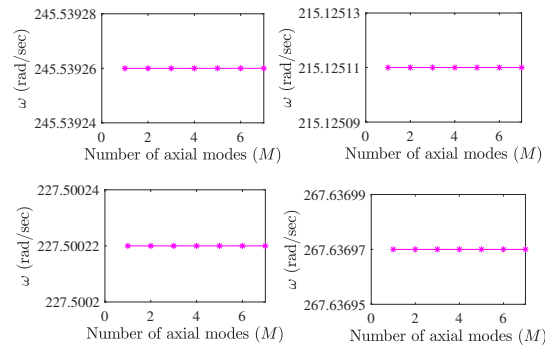


Figure 24: Convergence of natural frequencies with respect to number of axial modes chosen during Galerkin projection (BC-D). Here $E = 2 \times 10^{11}$ N/m², $\rho_m = 7680$ kg/m³, $\nu = 0.3$, $L = 2$ m, $R = 2$ m, and $h = 0.01$ m.

Appendix B: Stability Charts

B.1: BC- B

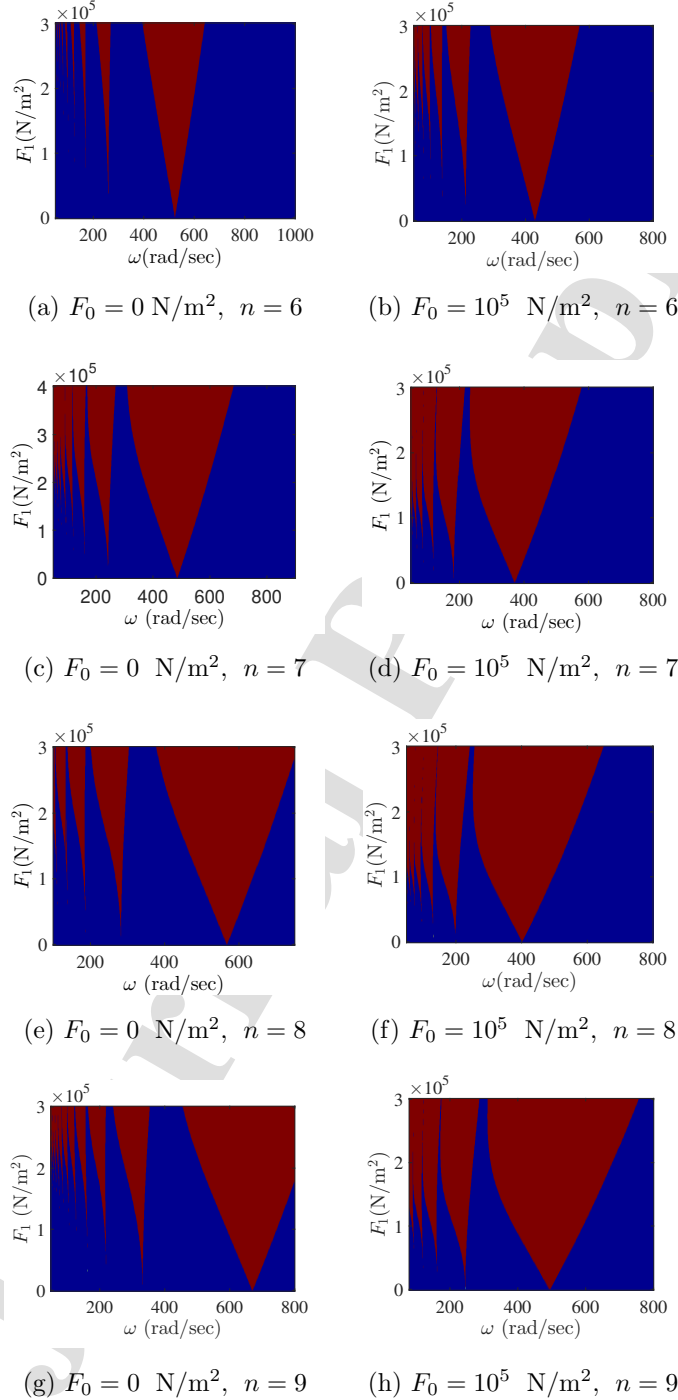


Figure 25: Stability charts for the circular cylindrical shell with pinned-free ends with axial constraint at the pinned end (BC-B) in the plane $F_1 - \omega$ at $F_0 = 0$ N/m² and $F_0 = 10^5$ N/m² for four circumferential wavenumbers $n = 6, 7, 8, 9$ and axial half wavenumber $m = 1$.

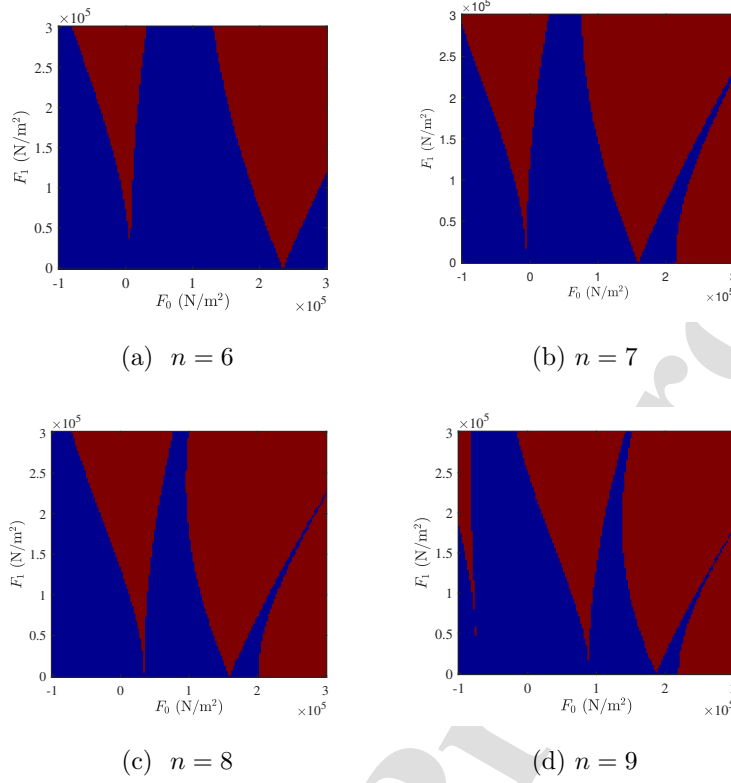


Figure 26: Stability charts for the circular cylindrical shell with pinned-free ends with axial constraint at the pinned end (BC-B) in the plane $F_0 - F_1$ at $\omega = 259$ (rad/sec) for different circumferential wavenumbers $n = 6, 7, 8, 9$ and axial half wavenumber $m = 1$.

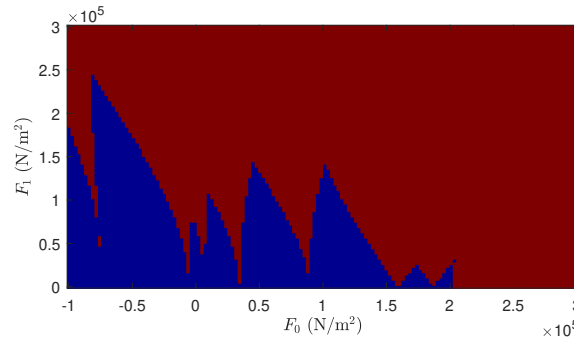


Figure 27: Combined stability charts for the pinned-free circular cylindrical shell with axial constraint at the pinned end (BC-B) in the plane $F_0 - F_1$ at $\omega = 259$ rad/sec for $n = 6, 7, 8, 9$ and $m = 1$.

B.2: BC-C

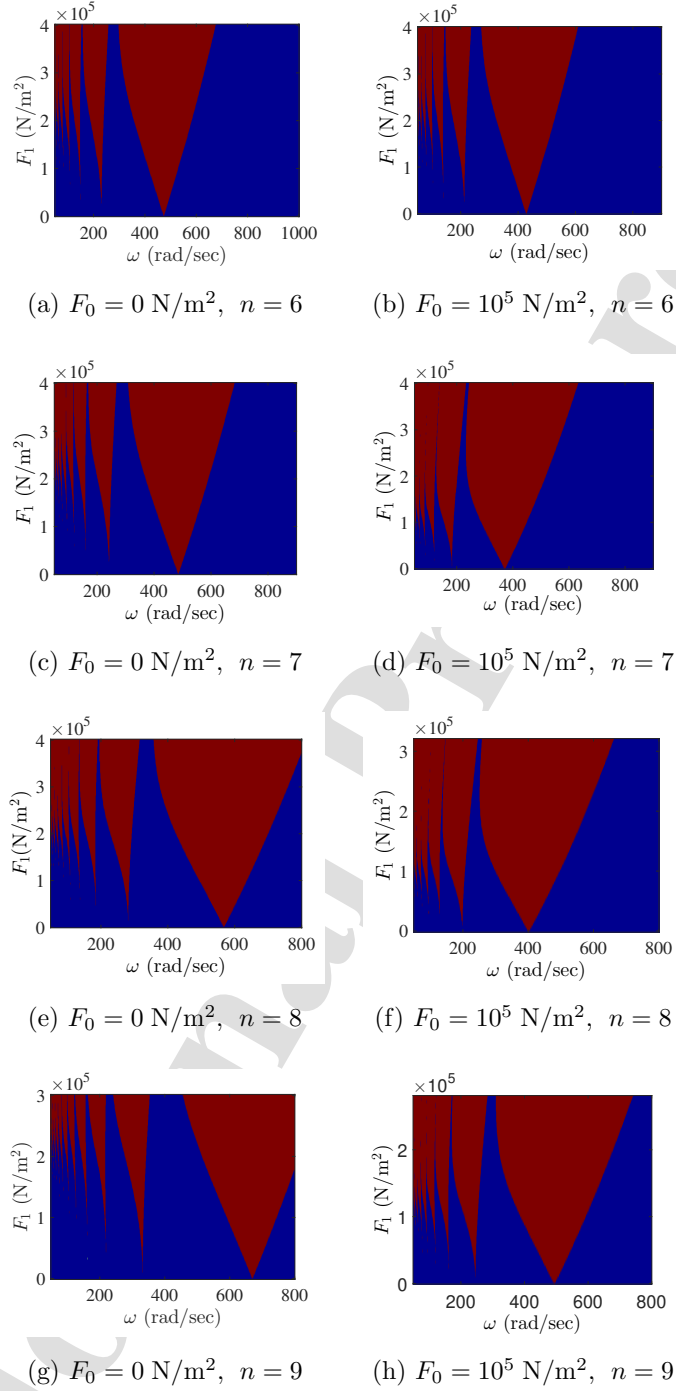


Figure 28: Stability charts for the circular cylindrical shell with fixed-free end conditions (BC-C) in the plane $F_1 - \omega$ at $F_0 = 0 \text{ N/m}^2$ (left column) and $F_0 = 10^5 \text{ N/m}^2$ (right column) for $n = 6, 7, 8, 9$ and $m = 1$.

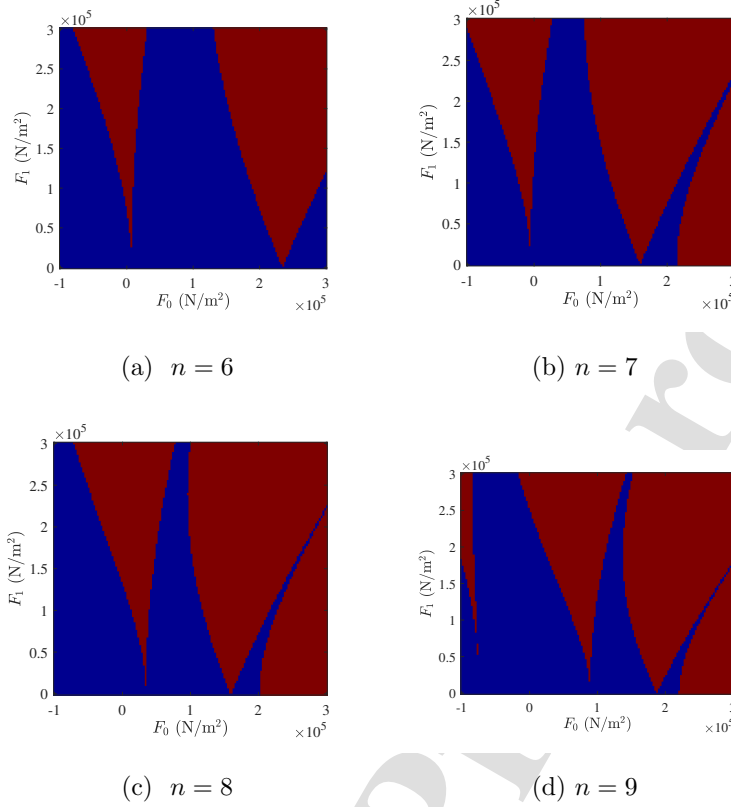


Figure 29: Stability charts for the circular cylindrical shell with fixed-free boundary conditions (BC-C) in the plane $F_0 - F_1$ at $\omega = 259$ (rad/sec) for different circumferential wavenumbers $n = 6, 7, 8, 9$ and axial half wavenumber $m = 1$.

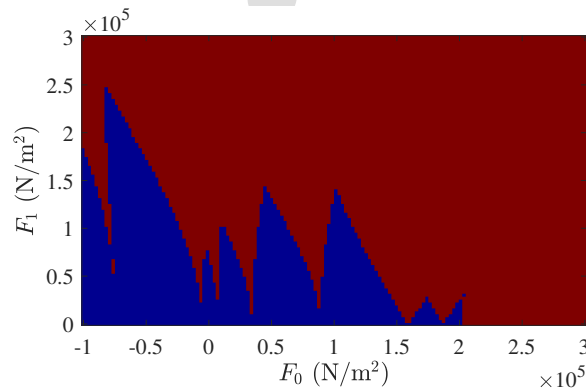


Figure 30: Combined stability chart for the circular cylindrical shell with fixed-free (BC-C) boundary conditions (BC-C) in the plane $F_0 - F_1$ at $\omega = 259$ rad/sec for $n = 6, 7, 8, 9$ and $m = 1$.

B.3: BC-D

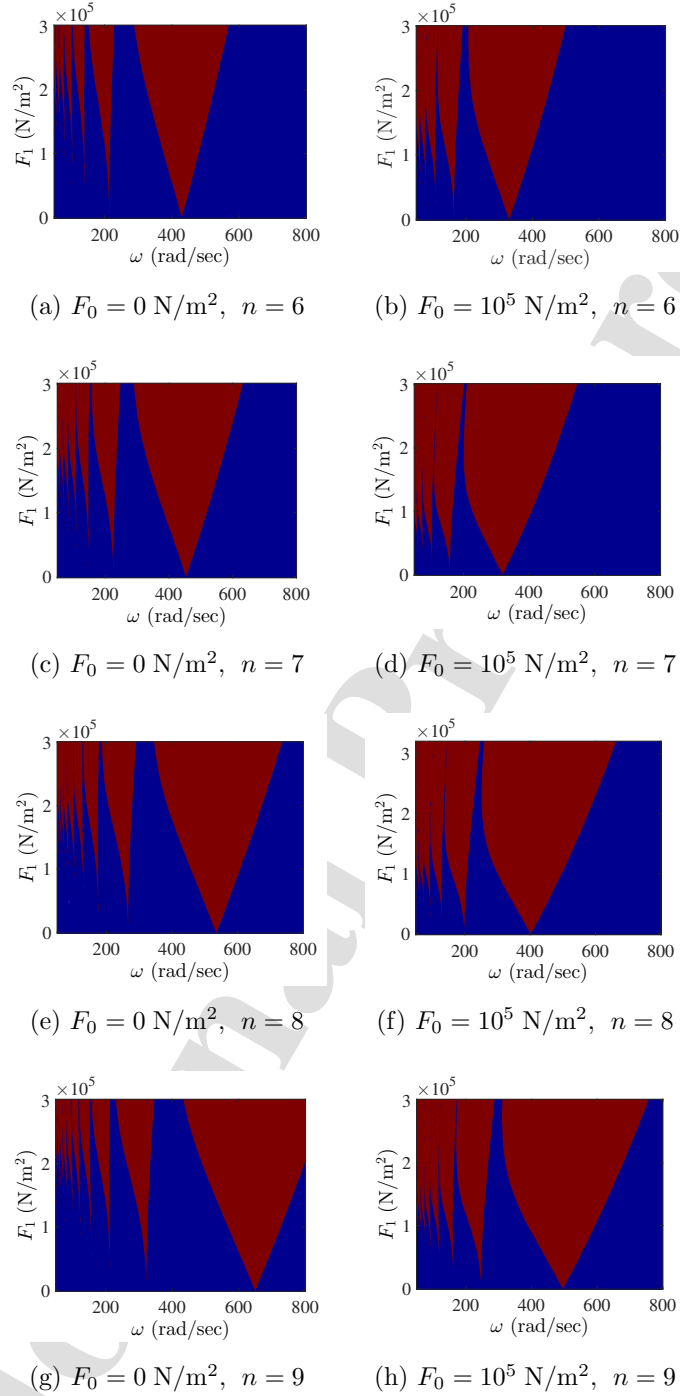


Figure 31: Combined stability charts for the pinned-free circular cylindrical shell with axial constraint at the free end (BC-D) in the planes $F_1 - \omega$ at $F_0 = 0 \text{ N/m}^2$ (left column) and $F_0 = 0.75 \times 10^5 \text{ N/m}^2$ (right column) for four consecutive circumferential wavenumbers $n = 8, 9, 10, 11$.

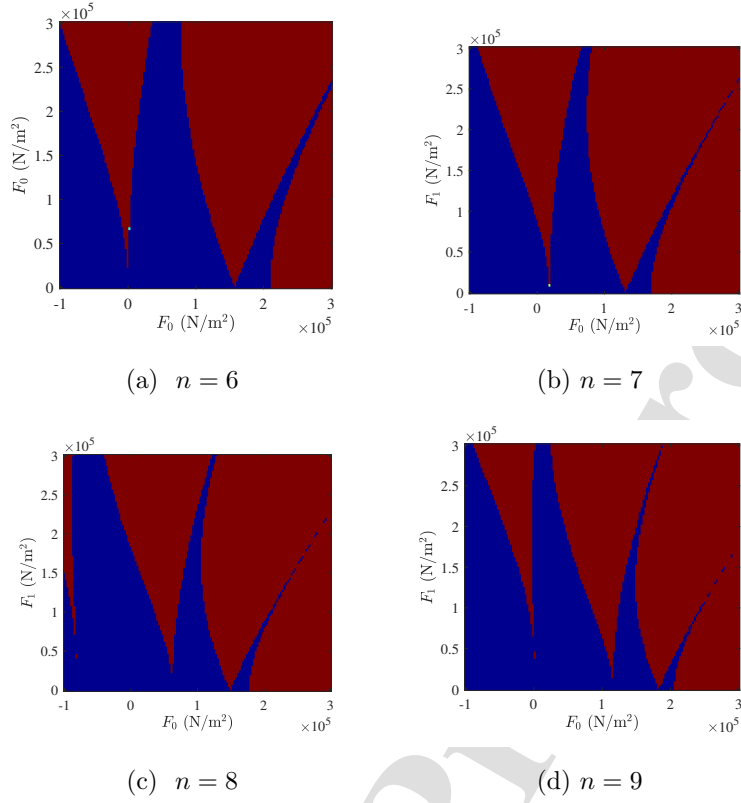


Figure 32: Combined stability charts for the pinned-free circular cylindrical shell with axial constraint at the free end (BC-D) in the plane $F_0 - F_1$ at a fixed value of excitation frequency $\omega = 215$ (rad/sec) for four consecutive circumferential wavenumbers $n = 6, 7, 8, 9$.

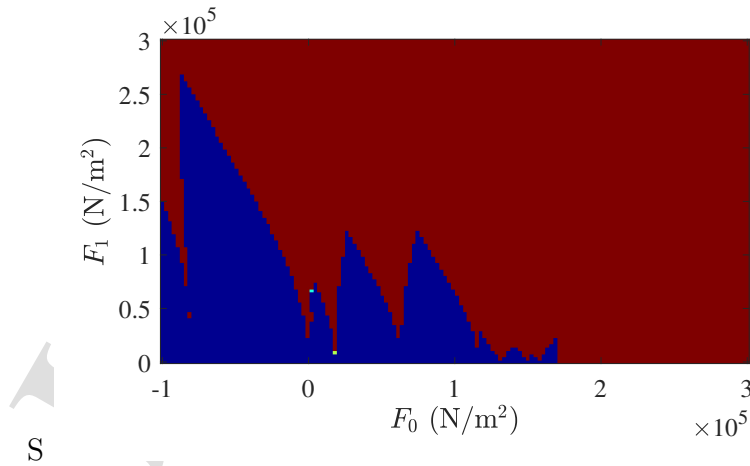


Figure 33: Combined stability chart for the pinned-free circular cylindrical shell with axial constraint at the free end (BC-D) in the plane $F_0 - F_1$ at $\omega = 215$ rad/sec for $n = 6, 7, 8, 9$ and $m = 1$.

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Highlights

- Parametric instability of cylindrical shells is investigated.
- Floquet theory is used to study higher dimensional Mathieu-Hill equations.
- Stability charts presented in the plane of forcing parameter.
- Stability analysis of cylindrical shells under static and dynamic loading
- Closed-form solution of the formulated BVP is obtained.
- Assumptions of Flügge-Lur'e-Byrne Theory of shells are applied

Credit Author Statement

Anish Kumar: Conceptualization, Methodology, Formal analysis, Validation, Writing- Reviewing and Editing **Pankaj Wahi.:** Conceptualization, Methodology, Reviewing and Editing, Supervision **Sovan Lal das :** Conceptualization, Methodology, Reviewing and Editing, Supervision **Krzysztof Kamil Zur:** Results Discussion, Reviewing and Editing,

Declaration of interests

☒ The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

☐ The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

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