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## Maximum entropy analysis of bulk arrival retrial queue with second optional service and Bernoulli vacation

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**Abstract:** This paper deals with the maximum entropy principle (MEP) to explore the steady state behaviour of the bulk arrival retrial queuing system. The concepts of Bernoulli vacation schedule and second optional service are taken into consideration. After the completion of first essential service (second optional service), the server has an option to go for vacation with probability  $\bar{p}(\bar{q})$  or may continue to serve the next customer, if any with complementary probability. During vacation, the server is allowed to do some secondary work and is called on working vacation. By introducing supplementary variables and using generating function technique, we obtain the expected number of the customers and expected waiting time of the customers in the retrial group. By employing the maximum entropy approach, we derive various system performance measures. By taking the numerical illustrations we perform a comparative study between the exact waiting time and the approximate waiting time based on MEP analysis. The sensitivity analysis is also carried out to validate the analytical results.

**Keywords:** bulk arrival; retrial queue; state dependent rate; supplementary variables; generating function; maximum entropy; queue size; waiting time.

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## 1 Introduction

Queueing system with retrial phenomenon is widely used to model stochastically many congestion situations encountered in client server communication (CSC), digital communication system (DCS), production system, carrier sense multiple access with collision detection (CSMA/CD) and many others. Any arriving batch finding the server busy or on working vacation enters a virtual pool of blocked customers called 'orbit'; otherwise one customer from the arriving batch receives the service immediately while the rest join the retrial group (orbit).

The motivation for studying a bulk arrival retrial queue with Bernoulli vacation and second optional service (SOS) comes from some numerous versatile applications encountered in production and manufacturing systems, computer and telecommunication networks, distribution and service sectors and many daily life queuing situations. For this purpose, we cite an example of maintenance department of production systems. Generally, the customer makes a call which may arrive in batches to the maintenance

department for repairing purpose. A customer may retry for his turn later on finding the maintenance department busy. On the other hand, if the maintenance department is free then one of the arriving calls is accepted while others are kept in the buffer. The repairman renders the first phase repair which is essential to restore the machine. If the caretaker of the machine is not satisfy then the repairman repairs again and provides second optional phase repair. When there is no call, the repairman may go for a vacation.

The present paper deals with  $M^X/(G_1, G_2)/1$  retrial queuing system under Bernoulli vacation schedule. Further, the present investigation includes the additional features, that is:

- 1 state dependent arrival rates
- 2 working vacation
- 3 SOS.

Moreover, the maximum entropy approach is employed to facilitate the explicit results for probabilities associated with the system states. The remainder of the paper is structured as follows. The survey of the previous relevant literature is presented in Section 2. Section 3 presents the description of the model with the help of the suitable assumptions and notations. In Section 4, the state governing equations are constructed by using the supplementary variable technique (SVT). In Section 5, the mean number of the customers in the orbit is derived. Various queuing characteristics have also been obtained. Some special cases have been deduced by setting appropriate parameters in Section 6. In Section 7, the principle of maximum entropy is described to establish the approximate results for the expected system size and expected waiting time. Numerical experiments and sensitivity analysis have been done in Section 8 to validate the analytic results. Finally, the conclusion is drawn in Section 9.

## 2 Literature review

Many researchers have studied the retrial queuing models in different frameworks. Chakravarthy and Dudin (2003) studied a retrial queuing model in which two types of customers arrive according to a Markovian arrival process. Wu et al. (2005) studied the retrial queues with general service times and non-exponential retrial time distribution. Sherman and Kharoufeh (2006) analysed an unreliable  $M/M/1$  retrial queue with infinite-capacity orbit. They provided the stability conditions as well as several stochastic decomposability results. Further, Wang et al. (2008) investigated an  $M/G/1$  retrial queuing system with disasters and unreliable server. They obtained the Laplace transforms of the transient solutions and also the steady-state solutions for both queuing and reliability measures of interest. The recent work on  $M/G/1$  retrial queue was done by Amandor and Artalejo (2009). They obtained the distribution of the successful and blocked events made by the primary customers and the retrial customers. Further, Boualem et al. (2009) considered an  $M/G/1$  retrial queue with vacations and derived several stochastic comparison properties for the stationary queue length distribution. Jain et al. (2012a) investigated a batch arrival retrial queuing system with essential and optional services with server breakdown. Recently,  $M^X/G/1$  retrial queue with unreliable server and general retrial times was examined by Bhagat and Jain (2013).

Queuing models with server vacations are characterised by the fact that the idle period of the server may be used for doing some other secondary work. Allowing the server to take vacations makes the queuing models more realistic and versatile for analysing the real-world congestion problems. Such types of applications occur in call centres, production systems, telecommunication networks, etc. In the queuing literature a variety of works has been done on queue with server vacation. In this connection, we refer the noble book by Tian and Zhang (2006). The comprehensive surveys on this topic can be found in Doshi (1986, 1990). Moreover, Choudhury and Madan (2004) considered a batch arrival queueing system, where the server provides two phases of heterogeneous service. They derived the queue size distribution at random epochs of the system and some important performance measures. Atencia and Moreno (2005) explored an  $M/G/1$  retrial queue with general retrial times with Bernoulli schedule. They obtained the generating function of the system size distribution and established the stochastic decomposition law. Choudhury et al. (2007) considered an  $M^X/G/1$  queue with two phases of heterogeneous service under Bernoulli vacation schedule under classical retrial policy. By employing the embedded Markov chain technique, they obtained the steady state distribution of the server state and number of the customers in the retrial group. Choudhury and Deka (2009) analysed the steady state behaviour of an  $M^X/G/1$  unreliable retrial queue with Bernoulli admission mechanism. Further, Sharma (2010) suggested threshold N-policy for un-reliable server queue with vacations. Working vacation queue with service interruption and multi-optional repair was investigated by Jain et al. (2011). Goswami and Selvaraju (2013) analysed a working vacation queueing model with priority customers where the customer's service times follow phase-type distributions.

In many realistic situations, all jobs demand the first 'essential' service, whereas only some of them demand the other 'optional' service. Wang (2004) studied an  $M/G/1$  queue with SOS and unreliable server. By using a supplementary variable method, they obtained the transient and steady-state solutions. A single server queue with two phases of heterogeneous service and linear retrial policy under Bernoulli vacation schedule was considered by Madan and Choudhury (2005). Choudhury and Paul (2006) studied a queueing model wherein the server provides two phases of heterogeneous service to each customer in succession with Bernoulli vacation schedule under different vacation policies. Further, Choudhury et al. (2007) examined the steady state behaviour of a batch arrival queue and Bernoulli schedule vacation under multiple vacation policy and obtained the queue size distribution of idle period process. The steady state behaviour of an  $M/G/1$  retrial queue with an additional second phase of optional service was examined by Choudhury and Deka (2008). Ke and Chang (2009) investigated a batch arrival retrial queue with general retrial times where the server provides two phases of heterogeneous service to all the customers under Bernoulli vacation schedules. Moreover, unreliable server queue with multi-optional repairs/services was studied by Jain et al. (2012d, 2013), respectively in different frameworks.

Maximum entropy approach is used by some researchers in the area of queueing theory to provide a reasonable and general approximation to the complex queueing systems. Jain and Dhakad (2003) provided the steady state queue size distribution for  $M^X/G/1$  queue by using the maximum entropy approach in which the constraints are expressed in terms of mean arrival rates, mean service rates and mean number of customers in the system. Ke and Lin (2006) studied the  $M^X/G/1$  queueing system with server vacations. They presented a comparative analysis between the approximate results with established exact results for various vacation time, service time and repair time

distributions by using the principle of maximum entropy. Wang et al. (2007) examined an  $M^X/M/1$  queuing system with multiple vacations. By using the maximum entropy approach, they developed the approximate formulas for the probability distributions of the number of customers in the system. Ke and Lin (2008) considered the  $M^X/G/1$  queuing system in which the server operates under N-policy and takes the single vacation. They used the maximum entropy principle (MEP) to derive the approximate formulas for the steady state probability distributions of the queue length. Wang and Huang (2009) analysed a single removable and unreliable server  $M/G/1$  queue under  $(p, N)$ -policy. They did the maximum entropy analysis to develop the approximate formulas for the probability distributions of the number of customers and the expected waiting time in the system. Jain et al. (2012b, 2012c) applied maximum entropy approach for un-reliable server queue in their study for different models.

### 3 Model description

In this section, we develop a retrial queuing system with SOS by incorporating the concepts of working vacation and Bernoulli vacation. For obtaining the approximate waiting time in terms of known operational measures, MEP is applied. In the previous study, these realistic assumptions are not taken into consideration together.

Consider a single server retrial queuing system with SOS. The server provides two phases of service i.e., the first essential service (FES) and SOS to all arriving customers. The essential service is needed to all the arriving customers whereas SOS is provided only to those customers who demand for the same. Now, we provide some assumptions to develop the mathematical model which are given as follows:

- The customers arrive in the system according to Poisson process with a fixed batch size. If  $X$  is the random variable denoting the batch size then batch size distribution is given by  $c_j = Pr[X = j]$ ,  $j = 1, 2, \dots, d$ . Further, the generating function for the batch size distribution is  $C(z) = \sum_{k=1}^{\infty} z^k c_k$ . Also, we note that  $C'(1) = E[X]$  and  $C''(1) = E[X^2]$ .
- As soon as the FES (SOS) of the customers is completed, the server may go for vacation with probability  $p$  ( $q$ ) or may continue to serve the next customer, if any with probability  $\bar{p}$  ( $\bar{q}$ ). After the completion of FES if the customer demands for the SOS, then server may provide the SOS with probability  $\sigma$  or becomes idle with probability  $\bar{\sigma}$ . We assume that FES and SOS are generally distributed with mean service times  $\frac{1}{\mu_1}$  and  $\frac{1}{\mu_2}$ .
- During working vacation period, the server may do some ancillary work; the service during vacation is general distributed with mean  $\frac{1}{\mu_3}$ .
- In retrial group, the time between the two successive attempts of each customer is considered to be exponentially distributed with rate  $\nu$ .

- Let  $N(t)$  and  $A(t)$  be the number of customers in the system and the server's state at time  $t$ . For the mathematical formulation of the model, we define the state of the server at time  $t$  as:

$$A(t) = \begin{cases} 0, & \text{if the server is in idle state} \\ 1, & \text{if the server is in FES state} \\ 2, & \text{if the server is in SOS state} \\ 3, & \text{if the server is on working vacation state} \end{cases}$$

- The state dependent arrival rates of the customers are given as follows:

$$\lambda_i = \begin{cases} \lambda_0, & \text{when the server is in idle state} \\ \lambda_1, & \text{when the server is in FES state} \\ \lambda_2, & \text{when the server is in SOS state} \\ \lambda_3, & \text{when the server is on working vacation state} \end{cases}$$

The notations used for general cumulative distribution function (CDF), probability distribution function (PDF), Laplace-Stieltjes transformation (LST) and the remaining service time (RST) or remaining working vacation time (RVT), respectively of FES, SOS and working vacation are given in Table 1.

**Table 1** Notations for distribution functions

| Time             | CDF      | PDF      | LST                   | RST/RVT    |
|------------------|----------|----------|-----------------------|------------|
| FES              | $S_1(x)$ | $s_1(x)$ | $\tilde{S}_1(\theta)$ | $S_1^0(x)$ |
| SOS              | $S_2(x)$ | $s_2(x)$ | $\tilde{S}_2(\theta)$ | $S_2^0(x)$ |
| Working vacation | $S_3(x)$ | $s_3(x)$ | $\tilde{S}_3(\theta)$ | $S_3^0(x)$ |

The steady state probabilities to construct the governing equations are defined as:  $P_{0,n}(t)dt = \Pr\{N(t) = n, A(t) = 0\}$ ,  $n \geq 0$  and  $P_{i,n}(x,t)dt = \Pr\{N(t) = n, A(t) = i, x \leq S_i^0(t) \leq x + dx\}$ ,  $n \geq 0, i = 1, 2, 3$ .

The  $r^{\text{th}}$  moment of FES time, SOS times and the working vacation time are denoted by  $E[S_1^r]$ ,  $E[S_2^r]$ , and  $E[S_3^r]$  where  $r \geq 1$ . Thus,  $E[S_1^r] = (-1)^r S_1^{(r)}(0)$ ;  $E[S_2^r] = (-1)^r S_2^{(r)}(0)$ ;  $E[S_3^r] = (-1)^r S_3^{(r)}(0)$ ;  $r \geq 1$ .

The Laplace transforms of probabilities  $P_{1,n}(x)$ ,  $P_{2,n}(x)$  and  $P_{3,n}(x)$  are denoted by

$$\tilde{P}_{1,n}(\theta), \tilde{P}_{2,n}(\theta) \text{ and } \tilde{P}_{3,n}(\theta) \text{ respectively, so that } \tilde{P}_{1,n}(\theta) = \int_0^\infty e^{-x\theta} P_{1,n}(x) dx, \tilde{P}_{2,n}(\theta) =$$

$$\int_0^\infty e^{-x\theta} P_{2,n}(x) dx \text{ and } \tilde{P}_{3,n}(\theta) = \int_0^\infty e^{-x\theta} P_{3,n}(x) dx.$$

#### 4 The analysis

SVT is one of the classical techniques which have been widely used to solve non-Markovian queuing model. To obtain the queue size distribution, we introduce the supplementary variables corresponding the elapsed time of general distributed random variables and then employ the probability generating function (*p.g.f.*) method. Now, we construct the Chapman Kolmogorov equations governing the model as follows:

$$(\lambda_0 + nv)P_{0,n} = P_{3,n}(0) + \bar{q}P_{2,n}(0) + (p\sigma + \bar{p}\bar{\sigma})P_{1,n}(0), n \geq 0 \quad (1)$$

$$\begin{aligned} -\frac{d}{dx}P_{1,n}(x) = & -\lambda_1 P_{1,n}(x) + \lambda_0 \sum_{k=1}^{n+1} c_k P_{0,n-k+1} S_1(x) \\ & + (n+1)vP_{0,n+1} S_1(x) + \lambda_1 \sum_{k=1}^n c_k P_{1,n-k}(x), n \geq 0 \end{aligned} \quad (2)$$

$$-\frac{d}{dx}P_{2,n}(x) = -\lambda_2 P_{2,n}(x) + \lambda_2 \sum_{k=1}^n c_k P_{2,n-k}(x) + \bar{p}\sigma P_{1,n}(0) S_2(x), n \geq 0 \quad (3)$$

$$-\frac{d}{dx}P_{3,n}(x) = -\lambda_3 P_{3,n}(x) + \lambda_3 \sum_{k=1}^n c_k P_{3,n-k}(x) + qP_{2,n}(0) S_3(x) + p\bar{\sigma}P_{1,n}(0) S_3(x), n \geq 0 \quad (4)$$

Define the probability generating functions as:

$$P_0(z) = \sum_{n=0}^{\infty} z^n P_{0,n} \quad (5)$$

$$\tilde{P}_i(z, \theta) = \sum_{n=0}^{\infty} z^n \tilde{P}_{i,n}(\theta), \quad i = 1, 2, 3 \quad (6)$$

$$P_i(z, 0) = \sum_{n=0}^{\infty} z^n P_{i,n}(0), \quad i = 1, 2, 3 \quad (7)$$

$$\tilde{P}_i(z, 0) = \sum_{n=0}^{\infty} z^n \tilde{P}_{i,n}(0), \quad i = 1, 2, 3 \quad (8)$$

*Lemma 1:* The partial probability generating functions for FES, SOS and working vacation respectively, are given by:

$$\tilde{P}_1(z, 0) = \frac{(\tilde{S}_1(\lambda_1 - \lambda_1 C(z) - 1) \left( \lambda_0 P_0(z) \frac{C(z)}{z} + vP'_0(z) \right))}{(\lambda_1 - \lambda_1 C(z))} \quad (9)$$

$$\tilde{P}_2(z, 0) = \frac{\bar{p}\sigma P_1(z, 0) (\tilde{S}_2(\lambda_2 - \lambda_2 C(z) - 1))}{(\lambda_2 - \lambda_2 C(z))} \quad (10)$$

$$\tilde{P}_3(z, 0) = \frac{(\tilde{S}_3(\lambda_3 - \lambda_3 C(z) - 1))(q\bar{p}\sigma\tilde{S}_2(\lambda_2 - \lambda_2 C(z)) + \bar{\sigma})R_1(z, 0)}{(\lambda_3 - \lambda_3 C(z))} \quad (11)$$

*Proof:* For proof see Appendix A.1

*Theorem 1:* The probability generating function of the number of the customers in the orbit is given by:

$$P(z) = \frac{P_0(z) \left[ \begin{aligned} &\psi - \lambda_0(\lambda_2\lambda_3(\tilde{S}_1(\lambda_1 - \lambda_1 C(z) - 1) \\ &+ \lambda_1\lambda_3\bar{p}\sigma\tilde{S}_1(\lambda_1 - \lambda_1 C(z))(\tilde{S}_2(\lambda_2 - \lambda_2 C(z) - 1) \\ &+ \lambda_1\lambda_2\tilde{S}_1(\lambda_1 - \lambda_1 C(z))(\tilde{S}_3(\lambda_3 - \lambda_3 C(z) - 1)(\bar{p}\sigma q\tilde{S}_2(\lambda_2 - \lambda_2 C(z)) + \bar{\sigma})) \end{aligned} \right]}{\psi} \quad (12)$$

where

$$\begin{aligned} P_0(z) &= P_0(1) \times \exp \left[ -\frac{\lambda_0}{v} \int_z^1 \frac{\left( 1 - [\tilde{S}_1(\lambda_1 - \lambda_1 C(u))(\bar{p}\sigma\tilde{S}_2(\lambda_2 - \lambda_2 C(u))(q\tilde{S}_3(\lambda_3 - \lambda_3 C(u)) + \bar{q}) + p\bar{\sigma}\tilde{S}_3(\lambda_3 - \lambda_3 C(u)) + (p\sigma + \bar{p}\bar{\sigma}))] \frac{C(u)}{u}}{[\tilde{S}_1(\lambda_1 - \lambda_1 C(u))(\bar{p}\sigma\tilde{S}_2(\lambda_2 - \lambda_2 C(u))(q\tilde{S}_3(\lambda_3 - \lambda_3 C(u)) + \bar{q}) + p\bar{\sigma}\tilde{S}_3(\lambda_3 - \lambda_3 C(u)) + (p\sigma + \bar{p}\bar{\sigma}))] - u} du \right] \\ P_0(1) &= \frac{1 - \rho}{[1 - \rho + \lambda_0 E[X][E[S_1] + \bar{p}\sigma E[S_2] + (\bar{p}\sigma q + \bar{\sigma})E[S_3]]]} \\ \psi &= \lambda_1\lambda_2\lambda_3 \left[ \tilde{S}_1(\lambda_1 - \lambda_1 C(z)) \left( \bar{p}\sigma\tilde{S}_2(\lambda_2 - \lambda_2 C(z))(q\tilde{S}_3(\lambda_3 - \lambda_3 C(z)) + \bar{q}) + p\bar{\sigma}\tilde{S}_3(\lambda_3 - \lambda_3 C(z)) + (p\sigma + \bar{p}\bar{\sigma}) \right) \right] \\ \rho &= \bar{p}\sigma E[X][\lambda_1 E[S_1] + \lambda_2 E[S_2] + q\lambda_3 E[S_3]] + p\bar{\sigma} E[X][\lambda_1 E[S_1] + \lambda_3 E[S_3]] \\ &\quad + (p\sigma + \bar{p}\bar{\sigma})\lambda_1 E[S_1] E[X] \end{aligned}$$

*Proof:* For proof see Appendix A.2.

## 5 Performance measures

In this section, we obtain some useful performance measures using the generating function results obtained in the previous section. These performance measures are helpful for the decision makers to analyse the concerned queuing model. The formulae for the long run probabilities of the server being idle, rendering FES/SOS and on working vacation state have been obtained explicitly. These results provide some meaningful interpretation of the concerned queuing model from application view point.

*Lemma 2:* The long run probabilities for which the server is in idle state, FES state, SOS state and in working vacation state are denoted by  $P(I)$ ,  $P(E)$ ,  $P(S)$  and  $P(V)$  respectively and are given by:

$$P(I) = P_0(1) \quad (13)$$

$$P(E) = \tilde{P}_1(1, 0) = \frac{\lambda_0 E[X]E[S_1]}{1 - \rho} P_0(1) \quad (14)$$

$$P(S) = \tilde{P}_2(1, 0) = \frac{\lambda_0 \bar{p}\sigma E[X]E[S_2]}{1 - \rho} P_0(1) \quad (15)$$

$$P(V) = \tilde{P}_3(1, 0) = \frac{\lambda_0 (\bar{p}\sigma q + \bar{\sigma}) E[X]E[S_3]}{1 - \rho} P_0(1) \quad (16)$$

*Proof:* For proof see Appendix A.3.

*Theorem 2:* The mean number of the customers in the orbit is:

$$E[N] = \left[ \frac{\lambda_0 (\rho + E[X] - 1)(\psi' - \lambda_0 \rho_1)}{v(1 - \rho)\psi'} + \frac{\psi'' - \lambda_0 \left( \eta_1 + \frac{E[X^2]\rho_1}{E[X]} \right)}{2\psi'} \right] P_0(1) - \frac{\psi''(\psi' - \lambda_0 \rho_1)}{2(\psi')^2} P_0(1) \quad (17)$$

where

$$\psi' = \lambda_1 \lambda_2 \lambda_3 (\rho - 1)$$

$$\psi'' = \lambda_1 \lambda_2 \lambda_3 \left( (E[X])^2 \eta + \frac{E[X^2]\rho}{E[X]} \right)$$

$$\eta = \bar{p}\sigma \left[ \lambda_1^2 E[S_1^2] + \lambda_2^2 E[S_2^2] + q\lambda_3^2 E[S_3^2] + 2\lambda_1 \lambda_2 E[S_1]E[S_2] \right] + 2q\lambda_1 \lambda_3 E[S_1]E[S_3] + 2q\lambda_2 \lambda_3 E[S_2]E[S_3]$$

$$p\bar{\sigma} \left[ \lambda_1^2 E[S_1^2] + \lambda_3^2 E[S_3^2] + 2\lambda_1 \lambda_3 E[S_1]E[S_3] \right] + \lambda_4^2 E[S_1^2] (p\sigma + \bar{p}\bar{\sigma})$$

$$\rho_1 = \lambda_1 \lambda_2 \lambda_3 E[X] [E[S_1] + \bar{p}\sigma E[S_2] + (\bar{p}\sigma q + \bar{\sigma}) E[S_3]]$$

$$\eta_1 = \lambda_1 \lambda_2 \lambda_3 \left[ \lambda_1 E[S_1^2] + \bar{p}\sigma \lambda_2 E[S_2^2] + (\bar{p}\sigma q + \bar{\sigma}) \lambda_3 E[S_3^2] + 2\lambda_1 \bar{p}\sigma E[S_1]E[S_2] + 2\lambda_1 (\bar{p}\sigma q + \bar{\sigma}) E[S_1]E[S_3] + 2\lambda_2 \bar{p}\sigma q E[S_3]E[S_2] \right]$$

*Proof:* For proof see Appendix A.4.

*Theorem 3:* The mean waiting time in the orbit is given by:

$$E[W] = \frac{\left[ \frac{\lambda_0(\rho + E[X] - 1)(\psi' - \lambda_0\rho_1)}{v(1-\rho)\psi'} + \frac{\psi'' - \lambda_0\left(\eta_1 + \frac{E[X^2]\rho_1}{E[X]}\right)}{2\psi'} \right] P_0(1) - \frac{\psi''(\psi' - \lambda_0\rho_1)}{2(\psi')^2} P_0(1)}{\lambda E[X]} \quad (18)$$

where

$$\lambda = \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3.$$

*Proof:* For proof see Appendix A.5.

## 6 Special cases

The model under consideration is quite general and allows to capture performance of more specific models which were developed previously by some researchers and available in queuing literature. In this section, we deduce results for some special cases by assigning the appropriate parameters to the system performance as follows:

### 6.1 Case I: $M^X/(G_1, G_2)/1$ retrial queue with two phases of essential service

In this case, we set  $\sigma = 1$ ,  $p = 0$  and  $\lambda_i = \lambda$ ,  $0 \leq i \leq 3$ , then equation (12) becomes:

$$P(z) = \frac{(1-z)P_0(z)}{[\tilde{S}_1(\lambda - \lambda C(z))\tilde{S}_2(\lambda - \lambda C(z))(q\tilde{S}_3(\lambda - \lambda C(z)) + \bar{q})] - z} \quad (19)$$

where

$$P_0(z) = P_0(1) \exp \left[ -\frac{\lambda}{v} \int_z^1 \frac{\left( 1 - [\tilde{S}_1(\lambda - \lambda C(u))\tilde{S}_2(\lambda - \lambda C(u))(q\tilde{S}_3(\lambda - \lambda C(u)) + \bar{q})] \frac{C(u)}{u} \right)}{[\tilde{S}_1(\lambda - \lambda C(u))\tilde{S}_2(\lambda - \lambda C(u))(q\tilde{S}_3(\lambda - \lambda C(u)) + \bar{q})] - u} du \right]$$

The above result coincides with the results obtained by Kumar and Arumuganathan (2008).

### 6.2 Case II: $M/G/1$ batch arrival retrial queue

In this case, substituting  $q$ ,  $\sigma = 1$ ,  $p = 0$ ,  $\tilde{S}_2(\lambda_2 - \lambda_2 C(z)) = 1$ ,  $\tilde{S}_3(\lambda_3 - \lambda_3 C(z)) = 1$  and  $\lambda_i = \lambda$ ,  $0 \leq i \leq 3$ , equation (12) yields:

$$P(z) = \frac{(1-z)P_0(z)}{[\tilde{S}_1(\lambda - \lambda C(z)) - z]} \quad (20)$$

where

$$P_0(z) = P_0(1) \exp \left[ -\frac{\lambda}{v} \int_z^1 \left( \frac{1 - [\tilde{S}_1(\lambda - \lambda C(u)) \frac{C(u)}{u}]}{[\tilde{S}_1(\lambda - \lambda C(u)) - u]} \right) du \right]$$

which coincides with the model studied by Falin (1976).

### 6.3 Case III: $M^X/(G_1, G_2)/1$ retrial queue with two phases of essential service and the Earlangian vacation time

Setting  $q, \sigma = 1, p = 0, \tilde{S}_2(\lambda_2 - \lambda_2 C(z)) = 1, \tilde{S}_3(\lambda_3 - \lambda_3 C(z)) = 1$  and  $\lambda_i = \lambda, 0 \leq i \leq 3$ , we get

$$\tilde{S}_3(\lambda - \lambda C(z)) = \left( \frac{\mu k}{\mu k + \lambda(1 - C(z))} \right)^k.$$

Now equation (12) becomes:

$$P(z) = \frac{(1-z)P_0(z)}{[\tilde{S}_1(\lambda - \lambda C(z)) \tilde{S}_2(\lambda - \lambda C(z)) (q \left( \frac{\mu k}{\mu k + \lambda(1 - C(z))} \right)^k + \bar{q})] - z} \quad (21)$$

where

$$P_0(z) = P_0(1) \exp \left[ -\frac{\lambda}{v} \int_z^1 \left( \frac{1 - [\tilde{S}_1(\lambda - \lambda C(u)) \tilde{S}_2(\lambda - \lambda C(u)) (q \left( \frac{\mu k}{\mu k + \lambda(1 - C(z))} \right)^k + \bar{q})] \frac{C(u)}{u}}{[\tilde{S}_1(\lambda - \lambda C(u)) \tilde{S}_2(\lambda - \lambda C(u)) (q \left( \frac{\mu k}{\mu k + \lambda(1 - C(z))} \right)^k + \bar{q})] - u} \right) du \right]$$

## 7 Maximum entropy approach

In this section, maximum entropy approach is used to obtain the steady state probabilities.  $M^X/(G_1, G_2)/1$  retrial queuing system with SOS under Bernoulli vacation schedule. Firstly, we provide the maximum entropy formalism by using several well known constraints. Further, by applying the method of Lagrange's multipliers we evaluate the Lagrange's function. The partial derivatives of the Lagrange's function with

respect to  $P_{0,n}$ ,  $P_{1,n}$ ,  $P_{2,n}$ ,  $P_{3,n}$  are put equal to zero. Now we formulate the maximum entropy model in order to obtain the steady state probabilities  $P_{0,n}$ ,  $P_{1,n}$ ,  $P_{2,n}$  and  $P_{3,n}$  in the following way:

### 7.1 The maximum entropy model

Following El-Affendi and Kouvatsos (1983), the entropy function  $H$  can be constructed as:

$$H = -\sum_{n=0}^{\infty} P_{0,n} \log P_{0,n} - \sum_{n=0}^{\infty} P_{1,n} \log P_{1,n} - \sum_{n=0}^{\infty} P_{2,n} \log P_{2,n} - \sum_{n=0}^{\infty} P_{3,n} \log P_{3,n} \quad (22)$$

subject to the following constraints:

- 1 normalising condition is:

$$\sum_{n=0}^{\infty} P_{0,n} + \sum_{n=0}^{\infty} P_{1,n} + \sum_{n=0}^{\infty} P_{2,n} + \sum_{n=0}^{\infty} P_{3,n} = 1 \quad (23)$$

- 2 the probability that the server is busy in providing FES is given by:

$$\sum_{n=0}^{\infty} P_{1,n} = \tilde{P}_1(1, 0) = a_1 \quad (24)$$

- 3 the probability that the server is rendering SOS is obtained as:

$$\sum_{n=0}^{\infty} P_{2,n} = \tilde{P}_2(1, 0) = a_2 \quad (25)$$

- 4 the probability that the server is on working vacation is:

$$\sum_{n=0}^{\infty} P_{3,n} = \tilde{P}_3(1, 0) = a_3 \quad (26)$$

- 5 the expected number of the customers in the system is:

$$\sum_{n=0}^{\infty} nP_{0,n} + \sum_{n=0}^{\infty} nP_{1,n} + \sum_{n=0}^{\infty} nP_{2,n} + \sum_{n=0}^{\infty} nP_{3,n} = E[N] \quad (27)$$

Here  $a_1$ ,  $a_2$ ,  $a_3$  and  $E[N]$  are known and can be determined by equations (14), (15), (16) and (17), respectively.

After introducing Lagrange's multipliers corresponding to the constraints (23)–(27), Lagrange's function is given by:

$$H = -\sum_{n=0}^{\infty} P_{0,n} \log P_{0,n} - \sum_{n=0}^{\infty} P_{1,n} \log P_{1,n} - \sum_{n=0}^{\infty} P_{2,n} \log P_{2,n} - \sum_{n=0}^{\infty} P_{3,n} \log P_{3,n} \\ - \theta_1 \left[ \sum_{n=0}^{\infty} P_{0,n} + \sum_{n=0}^{\infty} P_{1,n} + \sum_{n=0}^{\infty} P_{2,n} + \sum_{n=0}^{\infty} P_{3,n} - 1 \right] - \theta_2 \left[ \sum_{n=0}^{\infty} P_{1,n} - a_1 \right] \quad (28)$$

$$\begin{aligned}
& -\theta_3 \left[ \sum_{n=0}^{\infty} P_{2,n} - a_2 \right] - \theta_4 \left[ \sum_{n=0}^{\infty} P_{3,n} - a_3 \right] \\
& - \theta_5 \left[ \sum_{n=0}^{\infty} nP_{0,n} + \sum_{n=0}^{\infty} nP_{1,n} + \sum_{n=0}^{\infty} nP_{2,n} + \sum_{n=0}^{\infty} nP_{3,n} - E[N] \right]
\end{aligned}$$

where  $\theta_i$  ( $1 \leq i \leq 5$ ) are the Lagrange's multipliers corresponding to the constraints (23)–(27), respectively.

## 7.2 The maximum entropy solution

The maximum entropy solutions are obtained by taking the partial derivatives of  $H$  with respect to  $P_{0,n}$ ,  $P_{1,n}$ ,  $P_{2,n}$ ,  $P_{3,n}$ , respectively and setting the results equal to zero. Thus, we get:

$$\frac{\partial H}{\partial P_{0,n}} = -\log P_{0,n} - 1 - \theta_1 - \theta_5 n = 0 \quad (29)$$

$$\frac{\partial H}{\partial P_{1,n}} = -\log P_{1,n} - 1 - \theta_1 - \theta_2 - \theta_5 n = 0 \quad (30)$$

$$\frac{\partial H}{\partial P_{2,n}} = -\log P_{2,n} - 1 - \theta_1 - \theta_3 - \theta_5 n = 0 \quad (31)$$

$$\frac{\partial H}{\partial P_{3,n}} = -\log P_{3,n} - 1 - \theta_1 - \theta_4 - \theta_5 n = 0 \quad (32)$$

From equations (29)–(32), we obtain:

$$P_{0,n} = e^{-(1+\theta_1)} e^{-\theta_5 n}, \quad n \geq 0 \quad (33)$$

$$P_{1,n} = e^{-(1+\theta_1+\theta_2)} e^{-\theta_5 n}, \quad n \geq 0 \quad (34)$$

$$P_{2,n} = e^{-(1+\theta_1+\theta_3)} e^{-\theta_5 n}, \quad n \geq 0 \quad (35)$$

$$P_{3,n} = e^{-(1+\theta_1+\theta_4)} e^{-\theta_5 n}, \quad n \geq 0 \quad (36)$$

Let  $\zeta_1 = e^{-(1+\theta_1)}$ ,  $\zeta_i = e^{-\theta_i}$ ,  $2 \leq i \leq 5$ .

Then equations (33)–(36) reduce to:

$$P_{0,n} = \zeta_1 \zeta_5^n, \quad n \geq 0 \quad (37)$$

$$P_{1,n} = \zeta_1 \zeta_2 \zeta_5^n, \quad n \geq 0 \quad (38)$$

$$P_{2,n} = \zeta_1 \zeta_3 \zeta_5^n, \quad n \geq 0 \quad (39)$$

$$P_{3,n} = \zeta_1 \zeta_4 \zeta_5^n, \quad n \geq 0 \quad (40)$$

On substituting the values of  $P_{0,n}$ ,  $P_{1,n}$ ,  $P_{2,n}$  and  $P_{3,n}$  from equations (37)–(40) into equations (23)–(26), we have:

$$\sum_{n=0}^{\infty} \zeta_1 \zeta_5^n = \frac{\zeta_1}{1 - \zeta_5} = 1 - (a_1 + a_2 + a_3) \quad (41)$$

$$\sum_{n=0}^{\infty} \zeta_1 \zeta_2 \zeta_5^n = \frac{\zeta_1 \zeta_2}{1 - \zeta_5} = a_1 \quad (42)$$

$$\sum_{n=0}^{\infty} \zeta_1 \zeta_3 \zeta_5^n = \frac{\zeta_1 \zeta_3}{1 - \zeta_5} = a_2 \quad (43)$$

$$\sum_{n=0}^{\infty} \zeta_1 \zeta_4 \zeta_5^n = \frac{\zeta_1 \zeta_4}{1 - \zeta_5} = a_3 \quad (44)$$

It follows from equations (41)–(44) that:

$$\zeta_1 = (1 - \delta)(1 - \zeta_5) \quad (45)$$

$$\zeta_2 = \frac{a_1}{(1 - \delta)} \quad (46)$$

$$\zeta_3 = \frac{a_2}{(1 - \delta)} \quad (47)$$

$$\zeta_4 = \frac{a_3}{(1 - \delta)} \quad (48)$$

where  $\delta = a_1 + a_2 + a_3$ .

On substituting the values of  $\zeta_1$ ,  $\zeta_2$ ,  $\zeta_3$  and  $\zeta_4$  from equations (45)–(48) into equation (27) and after doing some algebraic manipulations, we obtain the maximum entropy approximate solutions. Now:

$$\zeta_5 = \frac{E[N]}{1 + E[N]} \quad (49)$$

On substituting the values of  $\zeta_1$ ,  $\zeta_2$ ,  $\zeta_3$  and  $\zeta_4$  from equations (45)–(48) into equations (37)–(40) and using equation (49), we finally get:

$$P_{0,n} = \left( \frac{1 - \delta}{1 + E[N]} \right) \left( \frac{E[N]}{1 + E[N]} \right)^n, \quad n \geq 0 \quad (50)$$

$$P_{1,n} = \left( \frac{a_1}{1 + E[N]} \right) \left( \frac{E[N]}{1 + E[N]} \right)^n, \quad n \geq 0 \quad (51)$$

$$P_{2,n} = \left( \frac{a_2}{1 + E[N]} \right) \left( \frac{E[N]}{1 + E[N]} \right)^n, \quad n \geq 0 \quad (52)$$

$$P_{3,n} = \left( \frac{a_3}{1 + E[N]} \right) \left( \frac{E[N]}{1 + E[N]} \right)^n, \quad n \geq 0 \quad (53)$$

*Theorem 4:* The approximate expected waiting time in the orbit.

The approximate expected waiting time in the orbit based on MEP is given by:

$$\begin{aligned}
 E[\hat{W}] = & \sum_{i=1}^2 \sum_{n=0}^{\infty} \left[ \frac{E[S_i]}{2} \left( \frac{E[X^2]}{E[X]} - 1 \right) \right] P_{0,n} \\
 & + \sum_{i=1}^2 \sum_{n=0}^{\infty} \left[ nE[S_i] + \frac{E[S_i]}{2} \left( \frac{E[X^2]}{E[X]} - 1 \right) \right] P_{i,n} \\
 & + \sum_{i=1}^2 \sum_{n=0}^{\infty} \left[ \frac{E[S_3^2]}{2E[S_3]} + nE[S_i] + \frac{E[S_i]}{2} \left( \frac{E[X^2]}{E[X]} - 1 \right) \right] P_{3,n}
 \end{aligned} \tag{54}$$

*Proof:* Let the tagged customer ‘C’ find  $n$  customers waiting in the queue in front of him to receive their service from the server. The server will be in any one of the following states: idle state ( $I$ ), busy state ( $B$ ) which includes both FES and SOS and working vacation ( $V$ ) respectively as follows.

- 1 *Idle state (I):* In this state, the server provides the service to the arriving batch immediately. So in this case, customer ‘C’ will serve until the customers precede in the same group will be served. In the idle state the expected waiting time is

$$\sum_{i=1}^2 \left[ \frac{E[S_i]}{2} \left( \frac{E[X^2]}{E[X]} - 1 \right) \right].$$

- 2 *Busy state (B):* In this state, arbitrary customer ‘C’ has to wait the service time of  $n$  customers who waits in front of him and plus the additional time of those customers preceding him in the same group. Thus the mean waiting time of customer ‘C’ at state  $B$  is

$$\sum_{i=1}^2 \left[ nE[S_i] + \frac{E[S_i]}{2} \left( \frac{E[X^2]}{E[X]} - 1 \right) \right].$$

- 3 *Working vacation state (V):* When the arbitrary customer ‘C’ finds that the server is on working vacation state, he must wait the remaining vacation time, plus the service time of those  $n$  customers waiting in the orbit and at last waiting time of those customers preceding him in the same group. In this state the mean waiting time of arbitrary customer ‘C’ is

$$\sum_{i=1}^2 \left[ \frac{E[S_3^2]}{2E[S_3]} + nE[S_i] + \frac{E[S_i]}{2} \left( \frac{E[X^2]}{E[X]} - 1 \right) \right].$$

Using the results (1)–(3) given above, we can obtain the approximate expected waiting time in the orbit given in the equation (54), where the values of  $P_{0,n}$ ,  $P_{1,n}$ ,  $P_{2,n}$  and  $P_{3,n}$  are given in equations (50)–(53), respectively.

## 8 Numerical illustration and sensitivity analysis

Numerical simulation is required to accurately and quickly analyse the effect of various system parameters on the performance indices which are often needed for examining the behaviour of a queuing system. To validate the analytical results derived in the earlier sections, numerical illustration is given. The coding of the program has been done in software MATLAB and numerical results obtained are displayed via tables and graphs in order to facilitate the sensitivity analysis which demonstrates the effect of various parameters on the system performance measures.

Now, we present numerical example of production systems which we have also mentioned in the first section. The customer makes a call which arrive in batches of fixed batch size  $k = 2$  with rate  $\lambda = 0.8$  to the maintenance department. On finding the maintenance department busy, he may retry again with rate  $\nu = 0.9$ . The repairman provides the FES repair of the machine with rate  $\mu_1 = 3$ . After providing FES, repairman has option to go for a vacation with probability  $p = 0.4$  or may continue SOS with probability  $q = 0.6$ . If someone opt for SOS repair, it is provided with rate  $\mu_2 = 1$ . If there is no call, the repairman may go for a vacation with probability  $\sigma = 0.7$ . The number of repaired machine is calculated using equations (1) – (4) as  $E[N] = 153.80$ .

The accuracy of the maximum entropy results is examined by giving the comparative analysis between the exact  $E[W]$  and approximate  $E[\hat{W}]$  results for  $M^X/(G_1, G_2)/1$  retrial queuing system under Bernoulli vacation schedule. Tables 2–3 is displayed to examine the effect of various parameters such as  $\lambda$  and  $\mu$  on the long run probabilities  $P(I)$ ,  $P(E)$ ,  $P(S)$  and  $P(V)$  for  $M^X/E_K/1$  model. For numerical computation purpose, we fix the default parameters as  $\lambda_1 = 0.8$ ,  $\lambda_2 = 0.8$ ,  $\lambda_3 = 0.7$ ,  $p = 0.6$ ,  $q = 0.5$ ,  $\sigma = 0.6$ ,  $\nu = 0.5$ ,  $\mu_\square = 5$ ,  $\mu_2 = 2$ ,  $\mu_3 = 3$  and  $k = 3$  for Tables 2–4 and Figures 1–2. We also set the default parameters for Tables 5–7 as  $\lambda_0 = 0.9$ ,  $\lambda_1 = 0.9$ ,  $\lambda_2 = 0.8$ ,  $\lambda_3 = 0.9$ ,  $\nu = 0.01$ ,  $\mu_\square = 5$ ,  $\mu_2 = 2$ ,  $\mu_3 = 3$ . We assume that the service time follows three types of service time distributions which are given as:

- 1 Erlangian distribution (ES)
- 2 deterministic distribution (DS)
- 3 gamma distribution (GS).

For different types of models, first and second moments of service times are determined as follows:

- 1 for  $M^X/E_K/1$  model:

$$\left[ E[S_i] = \frac{1}{\mu} \right] \text{ and } \left[ E[S_i^1] = \frac{(k+1)}{k\mu^2} \right], \quad i = 1, 2, 3.$$

- 2 for  $M^X/D/1$  model:

$$\left[ E[S_i] = \frac{1}{\mu} \right] \text{ and } \left[ E[S_i^1] = \frac{1}{\mu^2} \right], \quad i = 1, 2, 3.$$

3 for  $M^X/\gamma/1$  model:

$$\left[ E[S_i] = \frac{1}{\mu} \right] \text{ and } \left[ E[S_i^1] = \frac{k(k+1)}{k\mu^2} \right], \quad i = 1, 2, 3.$$

## 8.1 Long run probabilities

### 8.1.1 Effect of arrival rate and service rate on the long run probabilities

From Table 2, it is observed that as arrival rate  $\lambda$  increases  $P(I)$  decreases while there is an increment in  $P(E)$ ,  $P(S)$  and  $P(V)$  for both case 1:  $(p, q, \sigma) = (.7, .6, .5)$  and case 2:  $(p, q, \sigma) = (.6, .5, .6)$ . Table 3 depicts that the probabilities  $P(E)$ ,  $P(S)$  and  $P(V)$  show the decreasing trends with respect to the increasing values of  $\mu$ . On the contrary, the probability of the server being idle  $P(I)$  increases.

**Table 2** Long run probabilities of the server being in different states by varying  $\lambda$  for  $M^X/E_K/1$  model

| $\lambda$ | Case 1: $(p, q, \sigma) = (.7, .6, .5)$ |        |        |        | Case 2: $(p, q, \sigma) = (.6, .5, .6)$ |        |        |        |
|-----------|-----------------------------------------|--------|--------|--------|-----------------------------------------|--------|--------|--------|
|           | $P(I)$                                  | $P(E)$ | $P(S)$ | $P(V)$ | $P(I)$                                  | $P(E)$ | $P(S)$ | $P(V)$ |
| 0.500     | 0.670                                   | 0.139  | 0.052  | 0.116  | 0.654                                   | 0.140  | 0.084  | 0.121  |
| 0.600     | 0.609                                   | 0.165  | 0.062  | 0.137  | 0.590                                   | 0.166  | 0.099  | 0.143  |
| 0.700     | 0.550                                   | 0.190  | 0.071  | 0.158  | 0.528                                   | 0.191  | 0.114  | 0.165  |
| 0.800     | 0.492                                   | 0.215  | 0.080  | 0.179  | 0.467                                   | 0.215  | 0.129  | 0.187  |
| 0.900     | 0.436                                   | 0.238  | 0.089  | 0.199  | 0.408                                   | 0.239  | 0.143  | 0.207  |
| 1.00      | 0.382                                   | 0.262  | 0.098  | 0.218  | 0.350                                   | 0.263  | 0.157  | 0.228  |

**Table 3** Long run probabilities of the server being in different states by varying  $\mu$  for  $M^X/E_K/1$  model

| $\mu$ | Case 1: $(p, q, \sigma) = (.7, .6, .5)$ |        |        |        | Case 2: $(p, q, \sigma) = (.6, .5, .6)$ |        |        |        |
|-------|-----------------------------------------|--------|--------|--------|-----------------------------------------|--------|--------|--------|
|       | $P(I)$                                  | $P(E)$ | $P(S)$ | $P(V)$ | $P(I)$                                  | $P(E)$ | $P(S)$ | $P(V)$ |
| 2.00  | 0.061                                   | 0.539  | 0.080  | 0.269  | 0.044                                   | 0.543  | 0.130  | 0.282  |
| 3.00  | 0.329                                   | 0.385  | 0.057  | 0.192  | 0.318                                   | 0.387  | 0.092  | 0.201  |
| 4.00  | 0.478                                   | 0.299  | 0.044  | 0.149  | 0.470                                   | 0.300  | 0.072  | 0.156  |
| 5.00  | 0.573                                   | 0.245  | 0.036  | 0.122  | 0.566                                   | 0.246  | 0.059  | 0.127  |
| 6.00  | 0.638                                   | 0.207  | 0.031  | 0.103  | 0.633                                   | 0.208  | 0.049  | 0.108  |
| 7.00  | 0.686                                   | 0.179  | 0.026  | 0.089  | 0.682                                   | 0.180  | 0.043  | 0.093  |

## 8.2 Queue length and waiting time

### 8.2.1 Effect of arrival rate, service rate and probability on the average queue length and waiting time

It can be observed from Table 4 that the average queue length and waiting time increase (decrease) as we increase the values of parameters  $\lambda_1$ ,  $\sigma$  ( $\mu_1$ ,  $p$ ) for all the three models.

The numerical values of average queue length and waiting time is lesser for the  $M^X/D/1$  model in comparison to  $M^X/E_K/1$  and  $M^X/\gamma/1$  model.

**Table 4** Effect on average queue length and waiting time by varying  $\lambda_1$ ,  $\mu_1$ ,  $\sigma$  and  $p$  for (1)  $M^X/E_K/1$ , (2)  $M^X/D/1$  and (3)  $M^X/\gamma/1$  model

| $\lambda_1$ | $M^X/E_K/1$ |        | $M^X/D/1$ |        | $M^X/\gamma/1$ |        |
|-------------|-------------|--------|-----------|--------|----------------|--------|
|             | $E[N]$      | $E[W]$ | $E[N]$    | $E[W]$ | $E[N]$         | $E[W]$ |
| 0.8         | 180.10      | 79.45  | 179.95    | 79.39  | 181.45         | 80.05  |
| 0.82        | 183.37      | 80.42  | 183.21    | 80.35  | 184.75         | 81.03  |
| 0.84        | 186.71      | 81.41  | 186.55    | 81.34  | 188.13         | 82.03  |
| 0.86        | 190.13      | 82.43  | 189.97    | 82.36  | 191.59         | 83.06  |
| 0.88        | 193.64      | 83.46  | 193.48    | 83.39  | 195.14         | 84.11  |
| 0.9         | 197.24      | 84.53  | 197.07    | 84.46  | 198.77         | 85.19  |
| <hr/>       |             |        |           |        |                |        |
| $\mu_1$     | $E[N]$      | $E[W]$ | $E[N]$    | $E[W]$ | $E[N]$         | $E[W]$ |
| 2.0         | 197.24      | 84.53  | 197.07    | 84.46  | 198.77         | 85.19  |
| 2.02        | 196.38      | 84.16  | 196.21    | 84.09  | 197.90         | 84.81  |
| 2.04        | 195.54      | 83.80  | 195.37    | 83.73  | 197.05         | 84.45  |
| 2.06        | 194.72      | 83.45  | 194.55    | 83.38  | 196.22         | 84.09  |
| 2.08        | 193.92      | 83.11  | 193.76    | 83.04  | 195.41         | 83.74  |
| 2.1         | 193.14      | 82.77  | 192.98    | 82.70  | 194.62         | 83.40  |
| <hr/>       |             |        |           |        |                |        |
| $\sigma$    | $E[N]$      | $E[W]$ | $E[N]$    | $E[W]$ | $E[N]$         | $E[W]$ |
| 0.5         | 185.92      | 79.68  | 185.76    | 79.61  | 187.39         | 80.31  |
| 0.6         | 197.24      | 84.53  | 197.07    | 84.46  | 198.77         | 85.19  |
| 0.7         | 209.52      | 89.79  | 209.34    | 89.71  | 211.12         | 90.48  |
| 0.8         | 222.88      | 95.52  | 222.69    | 95.44  | 224.56         | 96.241 |
| 0.9         | 237.48      | 101.78 | 237.29    | 101.69 | 239.24         | 102.53 |
| 1.0         | 253.50      | 108.64 | 253.30    | 108.55 | 255.35         | 109.43 |
| <hr/>       |             |        |           |        |                |        |
| $p$         | $E[N]$      | $E[W]$ | $E[N]$    | $E[W]$ | $E[N]$         | $E[W]$ |
| 0.5         | 223.98      | 95.99  | 223.79    | 95.91  | 225.72         | 96.73  |
| 0.6         | 197.24      | 84.53  | 197.07    | 84.46  | 198.77         | 85.19  |
| 0.7         | 174.65      | 74.85  | 174.59    | 74.78  | 176.01         | 75.43  |
| 0.8         | 155.31      | 66.56  | 155.11    | 66.50  | 156.51         | 67.07  |
| 0.9         | 138.55      | 59.37  | 138.45    | 59.32  | 139.62         | 59.83  |
| 1.0         | 123.88      | 53.09  | 123.77    | 53.04  | 124.82         | 53.49  |

### 8.2.2 Effect of arrival rate, service rate and retrial rate on exact and approximate waiting time

Tables 5–7 demonstrate a comparison between the exact ( $E[W]$ ) and the approximate ( $E[\hat{W}]$ ) results of average waiting time for different types of models for case 1:  $(p, q, \sigma) = (.7, .8, .6)$  and case 2:  $(p, q, \sigma) = (.6, .5, .6)$ . Tables 5–7 show that  $E[W]$  and  $(E[\hat{W}])$  increase as we increase  $\lambda_1$  and  $\lambda_2$ . The decreasing trend of  $E[W]$  and  $(E[\hat{W}])$  with the increasing values of  $\mu_1$ ,  $\mu_2$ ,  $\nu$  for  $M^X/E_K/1$ ,  $M^X/\gamma/1$  and  $M^X/D/1$  models can also be seen.

The absolute percentage error (*APE*) in  $(E[\hat{W}])$  is calculated which varies from 0%–11% as noticed in Tables 5–7.

**Table 5** Comparison between the exact and the approximate results of waiting times for  $M^X/E_K/1$  model

| Case 1: $(p, q, \sigma) = (.7, .8, .6)$ |        |              |            | Case 2: $(p, q, \sigma) = (.6, .5, .6)$ |              |            |
|-----------------------------------------|--------|--------------|------------|-----------------------------------------|--------------|------------|
| $\lambda_1$                             | $E[W]$ | $E[\hat{W}]$ | <i>APE</i> | $E[W]$                                  | $E[\hat{W}]$ | <i>APE</i> |
| 0.800                                   | 84.7   | 77.1         | 9.0        | 79.4                                    | 72.6         | 8.6        |
| 0.820                                   | 85.8   | 78.9         | 8.0        | 80.4                                    | 74.2         | 7.6        |
| 0.840                                   | 86.9   | 80.8         | 7.0        | 81.4                                    | 76.0         | 6.6        |
| 0.860                                   | 88.0   | 82.7         | 6.0        | 82.4                                    | 77.7         | 5.6        |
| 0.880                                   | 89.1   | 84.7         | 4.9        | 83.4                                    | 79.6         | 4.6        |
| 0.900                                   | 90.3   | 86.8         | 3.9        | 84.5                                    | 81.4         | 3.6        |
| $\lambda_2$                             | $E[W]$ | $E[\hat{W}]$ | <i>APE</i> | $E[W]$                                  | $E[\hat{W}]$ | <i>APE</i> |
| 0.700                                   | 89.1   | 82.2         | 7.7        | 82.3                                    | 75.9         | 7.7        |
| 0.740                                   | 89.6   | 84.0         | 6.2        | 83.1                                    | 78.1         | 6.0        |
| 0.780                                   | 90.1   | 85.8         | 4.7        | 84.0                                    | 80.3         | 4.4        |
| 0.820                                   | 90.6   | 87.7         | 3.1        | 84.9                                    | 82.6         | 2.7        |
| 0.860                                   | 91.1   | 89.7         | 1.5        | 85.9                                    | 85.0         | 1.0        |
| 0.900                                   | 91.7   | 91.7         | 0.02       | 86.9                                    | 87.5         | 0.6        |
| $\mu_1$                                 | $E[W]$ | $E[\hat{W}]$ | <i>APE</i> | $E[W]$                                  | $E[\hat{W}]$ | <i>APE</i> |
| 2.00                                    | 90.3   | 86.8         | 3.9        | 84.5                                    | 81.4         | 3.6        |
| 2.02                                    | 90.0   | 85.7         | 4.7        | 84.1                                    | 80.3         | 4.4        |
| 2.04                                    | 89.7   | 84.7         | 5.5        | 83.8                                    | 79.3         | 5.3        |
| 2.06                                    | 89.4   | 83.7         | 6.4        | 83.4                                    | 78.2         | 6.2        |
| 2.08                                    | 89.1   | 82.7         | 7.1        | 83.1                                    | 77.2         | 7.0        |
| 2.10                                    | 88.9   | 81.8         | 7.9        | 82.7                                    | 76.2         | 7.8        |
| $\mu_2$                                 | $E[W]$ | $E[\hat{W}]$ | <i>APE</i> | $E[W]$                                  | $E[\hat{W}]$ | <i>APE</i> |
| 3.00                                    | 90.3   | 86.8         | 3.9        | 84.5                                    | 81.4         | 3.6        |
| 3.20                                    | 87.4   | 82.4         | 5.7        | 82.8                                    | 78.6         | 5.0        |
| 3.40                                    | 85.0   | 78.7         | 7.3        | 81.3                                    | 76.1         | 6.3        |
| 3.60                                    | 82.9   | 75.6         | 8.8        | 80.0                                    | 74.0         | 7.4        |
| 3.80                                    | 81.1   | 72.9         | 10.1       | 78.8                                    | 72.1         | 8.4        |
| 4.00                                    | 79.5   | 70.5         | 11.2       | 77.8                                    | 70.5         | 9.4        |
| $\nu$                                   | $E[W]$ | $E[\hat{W}]$ | <i>APE</i> | $E[W]$                                  | $E[\hat{W}]$ | <i>APE</i> |
| 0.010                                   | 90.3   | 86.8         | 3.9        | 84.5                                    | 81.43        | 3.6        |
| 0.016                                   | 57.2   | 55.4         | 3.2        | 53.6                                    | 52.0         | 2.5        |
| 0.022                                   | 42.2   | 41.1         | 2.5        | 39.5                                    | 38.7         | 2.1        |
| 0.028                                   | 33.6   | 33.0         | 1.8        | 31.5                                    | 31.0         | 1.4        |
| 0.034                                   | 28.1   | 27.7         | 1.2        | 26.3                                    | 26.1         | 0.7        |
| 0.040                                   | 24.2   | 24.0         | 0.5        | 22.6                                    | 22.6         | 0.05       |

**Table 6** Comparison between the exact and the approximate results of waiting times for  $M^X/\gamma/1$  model

| <i>Case 1: <math>(p, q, \sigma) = (.7, .8, .6)</math></i> |        |              |       | <i>Case 2: <math>(p, q, \sigma) = (.6, .5, .6)</math></i> |              |       |
|-----------------------------------------------------------|--------|--------------|-------|-----------------------------------------------------------|--------------|-------|
| $\lambda_1$                                               | $E[W]$ | $E[\hat{W}]$ | $APE$ | $E[W]$                                                    | $E[\hat{W}]$ | $APE$ |
| 0.800                                                     | 85.3   | 77.8         | 8.8   | 80.0                                                      | 73.3         | 8.4   |
| 0.820                                                     | 86.4   | 79.6         | 7.8   | 81.0                                                      | 74.9         | 7.4   |
| 0.840                                                     | 87.3   | 81.5         | 6.8   | 82.0                                                      | 76.7         | 6.4   |
| 0.860                                                     | 88.6   | 83.5         | 5.8   | 83.0                                                      | 78.5         | 5.4   |
| 0.880                                                     | 89.8   | 85.5         | 4.7   | 84.1                                                      | 80.3         | 4.4   |
| 0.900                                                     | 91.0   | 87.6         | 3.7   | 85.1                                                      | 82.2         | 3.4   |
| $\lambda_2$                                               | $E[W]$ | $E[\hat{W}]$ | $APE$ | $E[W]$                                                    | $E[\hat{W}]$ | $APE$ |
| 0.700                                                     | 89.7   | 82.9         | 7.5   | 82.9                                                      | 76.6         | 7.5   |
| 0.740                                                     | 90.2   | 84.7         | 6.0   | 83.8                                                      | 78.4         | 5.9   |
| 0.780                                                     | 90.7   | 86.6         | 4.5   | 84.7                                                      | 81.1         | 4.2   |
| 0.820                                                     | 91.3   | 88.5         | 2.9   | 85.6                                                      | 83.4         | 2.6   |
| 0.860                                                     | 91.8   | 90.5         | 1.4   | 86.6                                                      | 85.8         | 0.9   |
| 0.900                                                     | 92.4   | 92.6         | 0.1   | 87.6                                                      | 88.3         | 0.8   |
| $\mu_1$                                                   | $E[W]$ | $E[\hat{W}]$ | $APE$ | $E[W]$                                                    | $E[\hat{W}]$ | $APE$ |
| 2.00                                                      | 91.0   | 87.6         | 3.7   | 85.1                                                      | 82.2         | 3.4   |
| 2.02                                                      | 90.7   | 86.5         | 4.6   | 84.8                                                      | 81.1         | 4.3   |
| 2.04                                                      | 90.4   | 85.5         | 5.4   | 84.4                                                      | 80.0         | 5.1   |
| 2.06                                                      | 90.1   | 84.5         | 6.2   | 84.0                                                      | 79.0         | 6.0   |
| 2.08                                                      | 89.8   | 83.5         | 7.0   | 83.7                                                      | 77.9         | 6.8   |
| 2.10                                                      | 89.5   | 82.5         | 7.7   | 83.4                                                      | 77.0         | 7.6   |
| $\mu_2$                                                   | $E[W]$ | $E[\hat{W}]$ | $APE$ | $E[W]$                                                    | $E[\hat{W}]$ | $APE$ |
| 3.00                                                      | 91.0   | 87.6         | 3.7   | 85.1                                                      | 82.2         | 3.4   |
| 3.20                                                      | 88.1   | 83.1         | 5.5   | 83.4                                                      | 79.3         | 4.8   |
| 3.40                                                      | 85.6   | 79.4         | 7.2   | 81.9                                                      | 76.8         | 6.1   |
| 3.60                                                      | 83.5   | 76.2         | 8.6   | 80.5                                                      | 74.6         | 7.3   |
| 3.80                                                      | 81.6   | 73.5         | 9.9   | 79.4                                                      | 72.7         | 8.3   |
| 4.00                                                      | 80.0   | 71.1         | 11.1  | 78.4                                                      | 71.1         | 9.3   |
| $\nu$                                                     | $E[W]$ | $E[\hat{W}]$ | $APE$ | $E[W]$                                                    | $E[\hat{W}]$ | $APE$ |
| 0.010                                                     | 91.0   | 87.6         | 3.7   | 85.1                                                      | 82.2         | 3.4   |
| 0.016                                                     | 57.9   | 56.2         | 2.9   | 54.2                                                      | 52.8         | 2.6   |
| 0.022                                                     | 42.9   | 41.9         | 2.1   | 40.2                                                      | 39.4         | 1.7   |
| 0.028                                                     | 34.3   | 33.8         | 1.4   | 32.1                                                      | 31.8         | 1.0   |
| 0.034                                                     | 28.7   | 28.5         | 0.7   | 26.9                                                      | 26.9         | 0.2   |
| 0.040                                                     | 24.8   | 24.8         | 0.4   | 23.3                                                      | 23.4         | 0.4   |

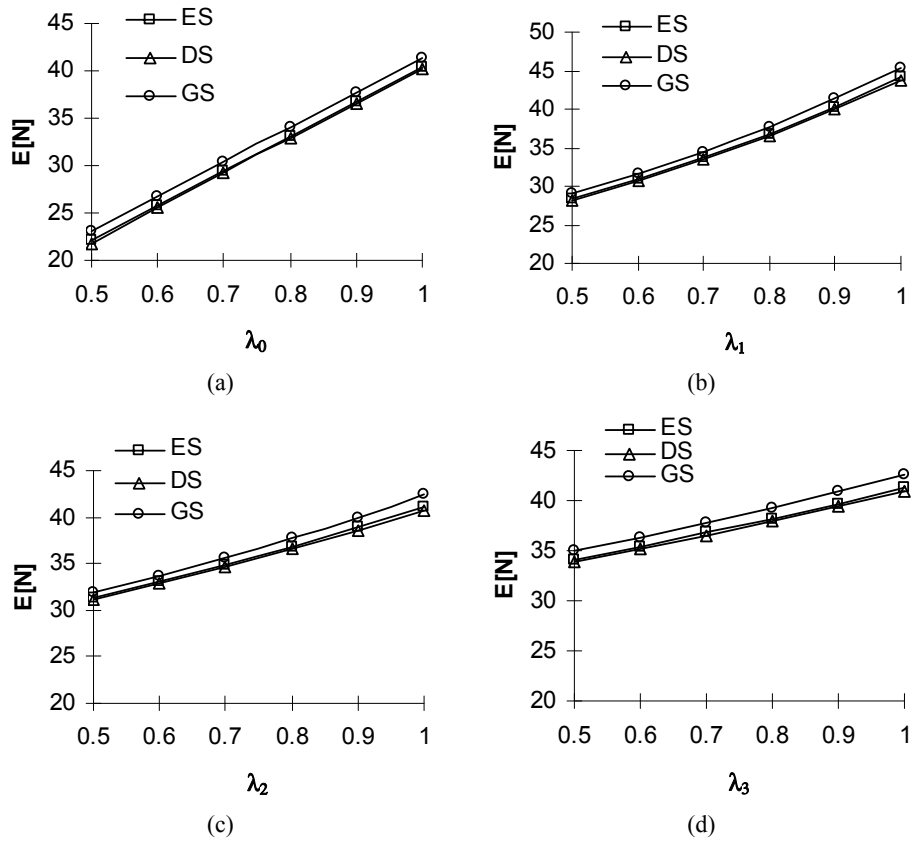
**Table 7** Comparison between the exact and the approximate results of waiting times for  $M^X/D/1$  model

| Case 1: $(p, q, \sigma) = (.7, .8, .6)$ |        |              |       | Case 2: $(p, q, \sigma) = (.6, .5, .6)$ |              |       |
|-----------------------------------------|--------|--------------|-------|-----------------------------------------|--------------|-------|
| $\lambda_1$                             | $E[W]$ | $E[\hat{W}]$ | $APE$ | $E[W]$                                  | $E[\hat{W}]$ | $APE$ |
| 0.800                                   | 84.6   | 77.0         | 9.0   | 79.3                                    | 72.5         | 8.6   |
| 0.820                                   | 85.7   | 78.8         | 8.0   | 80.3                                    | 74.2         | 7.6   |
| 0.840                                   | 86.8   | 80.7         | 7.0   | 81.3                                    | 75.9         | 6.6   |
| 0.860                                   | 87.9   | 82.6         | 6.0   | 82.3                                    | 77.6         | 5.6   |
| 0.880                                   | 89.1   | 84.6         | 4.9   | 83.3                                    | 79.5         | 4.6   |
| 0.900                                   | 90.2   | 86.7         | 3.9   | 84.4                                    | 81.3         | 3.6   |
| $\lambda_2$                             | $E[W]$ | $E[\hat{W}]$ | $APE$ | $E[W]$                                  | $E[\hat{W}]$ | $APE$ |
| 0.700                                   | 89.0   | 82.1         | 7.7   | 82.2                                    | 75.9         | 7.7   |
| 0.740                                   | 89.5   | 83.9         | 6.2   | 83.1                                    | 78.0         | 6.1   |
| 0.780                                   | 90.0   | 85.7         | 4.7   | 84.0                                    | 80.2         | 4.4   |
| 0.820                                   | 90.5   | 87.6         | 3.1   | 84.9                                    | 82.5         | 2.7   |
| 0.860                                   | 91.1   | 89.6         | 1.6   | 85.8                                    | 84.9         | 1.0   |
| 0.900                                   | 90.2   | 86.7         | 3.9   | 84.4                                    | 81.3         | 3.6   |
| $\mu_1$                                 | $E[W]$ | $E[\hat{W}]$ | $APE$ | $E[W]$                                  | $E[\hat{W}]$ | $APE$ |
| 2.00                                    | 90.2   | 86.7         | 3.9   | 84.4                                    | 81.3         | 3.6   |
| 2.02                                    | 89.9   | 85.6         | 4.7   | 84.0                                    | 80.2         | 4.5   |
| 2.04                                    | 89.6   | 84.6         | 5.6   | 83.7                                    | 79.2         | 5.3   |
| 2.06                                    | 89.4   | 83.6         | 6.4   | 83.3                                    | 78.1         | 6.2   |
| 2.08                                    | 89.1   | 82.6         | 7.2   | 83.0                                    | 77.1         | 7.0   |
| 2.10                                    | 88.8   | 81.7         | 7.9   | 82.7                                    | 76.1         | 7.8   |
| $\mu_2$                                 | $E[W]$ | $E[\hat{W}]$ | $APE$ | $E[W]$                                  | $E[\hat{W}]$ | $APE$ |
| 3.00                                    | 90.2   | 86.7         | 3.9   | 84.4                                    | 81.3         | 3.6   |
| 3.20                                    | 87.4   | 82.3         | 5.7   | 82.7                                    | 78.5         | 5.0   |
| 3.40                                    | 84.9   | 78.7         | 7.3   | 81.2                                    | 76.1         | 6.3   |
| 3.60                                    | 82.8   | 75.5         | 8.8   | 79.9                                    | 73.9         | 7.4   |
| 3.80                                    | 81.0   | 72.8         | 10.1  | 78.8                                    | 72.1         | 8.4   |
| 4.00                                    | 9.4    | 70.5         | 11.3  | 77.8                                    | 70.4         | 9.4   |
| $\nu$                                   | $E[W]$ | $E[\hat{W}]$ | $APE$ | $E[W]$                                  | $E[\hat{W}]$ | $APE$ |
| 0.010                                   | 90.2   | 86.7         | 3.9   | 84.4                                    | 81.3         | 3.6   |
| 0.016                                   | 57.2   | 55.3         | 3.2   | 53.5                                    | 51.9         | 2.8   |
| 0.022                                   | 42.1   | 41.0         | 2.5   | 39.4                                    | 38.6         | 2.1   |
| 0.028                                   | 33.5   | 32.9         | 1.9   | 31.4                                    | 30.9         | 1.4   |
| 0.034                                   | 28.0   | 27.6         | 1.2   | 26.2                                    | 26.0         | 0.7   |

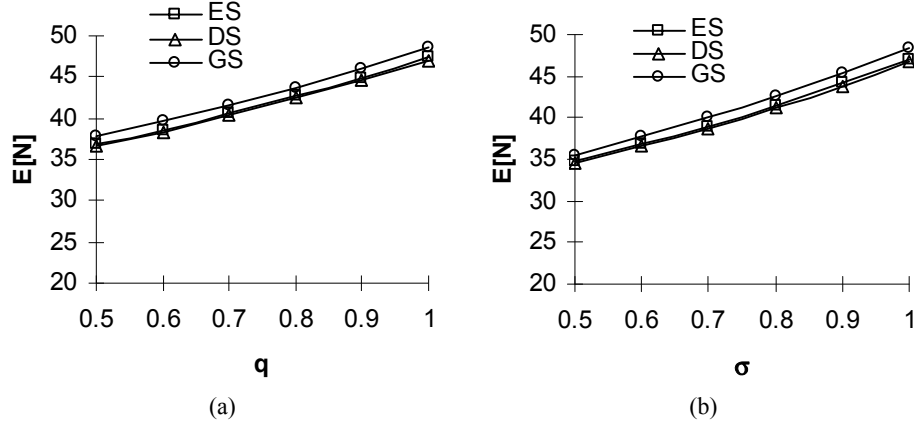
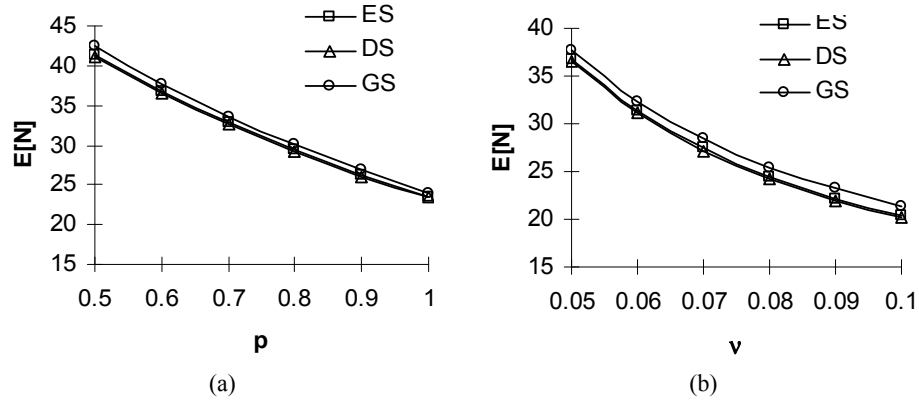
### 8.2.3 Graphical representation for the mean orbit size by varying arrival rate, retrial rate and probabilities

Figures 1(a) to 1(d) display the result for mean orbit size  $E[N]$  by varying the values of  $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ , for the different sets of Erlangian, deterministic and gamma service time distributions, respectively. In these figures, we observe that  $E[N]$  increases with the increasing values of  $\lambda_0, \lambda_1, \lambda_2, \lambda_3$  and this increment is higher for GS in comparison to ES and DS.

**Figure 1**  $E[N]$  vs (a)  $\lambda_0$ , (b)  $\lambda_1$ , (c)  $\lambda_2$  and (d)  $\lambda_3$  for different service time distributions



For the different sets of service time distributions, the mean orbit size ( $E[N]$ ) is compared with varying values of  $q, \sigma, p$  and  $\nu$  in Figures 2(a) to 2(b) and 3(a) to 3(b), respectively. It is clear from Figures 2(a) to 2(b) that  $E[N]$  exhibits an increasing trend as we increase the values of  $q, \sigma$ . We observe that  $E[N]$  increases sharply in case of GS. Figures 3(a) to 3(b) depict the decreasing pattern of mean orbit size for the increasing values of  $p$  and  $\nu$  the effect in case of GS is more prominent than ES and DS.

**Figure 2**  $E[n]$  vs (a)  $q$  and (b)  $\sigma$ , for different service time distributions**Figure 3**  $E[n]$  vs (a)  $p$  and (b)  $v$ , for different service time distributions

Based on numerical experiment performed, overall we conclude that:

- the varying trends in mean orbit size are more perceptible for gamma service time distributions as compared to Erlangian and deterministic service time distributions
- the exact and the approximate waiting times of the customers in the orbit increase with the increase in arrival rate but decrease with the increase in service rate and retrial rate
- based on the numerical results, we conclude that the maximum entropy approach provides a reasonable good approximation method for obtaining the probability distribution explicitly which is not easy to obtain by using classical method.

## 9 Conclusions

The performance analysis of bulk arrival retrial queue with essential and optional services under Bernoulli vacation schedule is presented. The concept of retrials of the customers for their service has wide applicability in many real life congestion situations including those in computer and communication networks, production and manufacturing systems, service and distribution systems, etc. It is worth mentioning that we have incorporated the concepts of bulk arrival, retrial queue, essential and optional services and Bernoulli vacation simultaneously which provide a more general model to be implemented in real time systems. Some special cases deduced exhibits the validity of our proposed model. We have evaluated the key performance indices such as system length, orbit length, waiting time, etc., and further provided numerical simulation and sensitivity analysis for three different types of service time distributions such as ES, DS and GS, which may be helpful to the system designers and decision makers for the quantitative assessment of concerned queuing system. By numerical illustration, it is established that MEP analysis approach provides a good accuracy. The research on the present investigation can be further extended by including the concepts of server breakdown, multi-optional service but the analysis will become more cumbersome.

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## Appendix

### A.1 Proof of Lemma 1

Taking the LST of equations (2)–(4), we have:

$$\begin{aligned} \theta \tilde{P}_{1,n}(\theta) - P_{1,n}(0) &= \lambda_1 \tilde{P}_{1,n}(\theta) - \lambda_0 \sum_{k=1}^{n+1} c_k \tilde{P}_{0,n-k+1}(\theta) - (n+1)vP_{0,n+1}(\theta) \tilde{S}(\theta) \\ &\quad - \lambda_1 \sum_{k=1}^n c_k \tilde{P}_{1,n-k}(\theta) \end{aligned} \quad (\text{A.1})$$

$$\theta \tilde{P}_{2,n}(\theta) - P_{2,n}(0) = \lambda_2 \tilde{P}_{2,n}(\theta) - \lambda_2 \sum_{k=1}^n c_k \tilde{P}_{2,n-k}(\theta) - \bar{p}\sigma P_{1,n}(0) \tilde{S}_2(\theta) \quad (\text{A.2})$$

$$\begin{aligned} \theta \tilde{P}_{3,n}(\theta) - P_{3,n}(0) &= \lambda_3 \tilde{P}_{3,n}(\theta) - \lambda_3 \sum_{k=1}^n c_k \tilde{P}_{3,n-k}(\theta) - qP_{2,n}(0) \tilde{S}_3(x) \\ &\quad - p\bar{\sigma}P_{1,n}(0) \tilde{S}_3(\theta) \end{aligned} \quad (\text{A.3})$$

On multiplying equations (1) and (A.1)–(A.3) by appropriate powers of  $z$  and then summing over  $n$ , we get:

$$\lambda_0 P_0(z) + vzP_0'(z) = P_3(z, 0) + \bar{q}P_2(z, 0) + (p\sigma + \bar{p}\bar{\sigma})P_1(z, 0) \quad (\text{A.4})$$

$$(\theta - \lambda_1 + \lambda_1 C(z)) \tilde{P}_1(z, \theta) = P_1(z, 0) - \frac{\lambda_0 C(z)}{z} P_0(z) \tilde{S}_1(\theta) - vP_0'(z) \tilde{S}_1(\theta) \quad (\text{A.5})$$

$$(\theta - \lambda_2 + \lambda_2 C(z)) \tilde{P}_2(z, \theta) = P_2(z, 0) - \bar{p}\sigma P_1(z, 0) \tilde{S}_2(\theta) \quad (\text{A.6})$$

$$(\theta - \lambda_3 + \lambda_3 C(z)) \tilde{P}_3(z, \theta) = P_3(z, 0) - qP_2(z, 0) \tilde{S}_3(\theta) - p\bar{\sigma}P_1(z, 0) \tilde{S}_3(\theta) \quad (\text{A.7})$$

Substituting  $\theta = \lambda_1 - \lambda_1 C(z)$ ,  $\lambda_2 C(z)$ ,  $\lambda_3 - \lambda_3 C(z)$  in equations (A.5)–(A.7), respectively, we have:

$$P_1(z, 0) = \frac{\lambda_0 C(z)}{z} P_0(z) \tilde{S}_1(\lambda_1 - \lambda_1 C(z)) + \nu P'_0(z) \tilde{S}_1(\lambda_1 - \lambda_1 C(z)) \quad (\text{A.8})$$

$$P_2(z, 0) = \bar{p}\sigma P_1(z, 0) \tilde{S}_2(\lambda_2 - \lambda_2 C(z)) \quad (\text{A.9})$$

$$P_3(z, 0) = qP_2(z, 0) \tilde{S}_3(\lambda_3 - \lambda_3 C(z)) + p\bar{\sigma}P_1(z, 0) \tilde{S}_3(\lambda_3 - \lambda_3 C(z)) \quad (\text{A.10})$$

Substituting the values of  $P_1(z, 0)$ ,  $P_2(z, 0)$  and  $P_3(z, 0)$  into equation (A.4), we get:

$$\begin{aligned} \lambda_0 P'_0(z) + \nu z P'_0(z) = & \left[ \bar{p}\sigma \tilde{S}_2(\lambda_2 - \lambda_2 C(z)) (q \tilde{S}_3(\lambda_3 - \lambda_3 C(z)) + \bar{q}) \right. \\ & \left. + p\bar{\sigma} \tilde{S}_3(\lambda_3 - \lambda_3 C(z)) + (p\sigma + \bar{p}\bar{\sigma}) \right] \\ & \times \left[ \frac{\lambda_0 C(z)}{z} P_0(z) + \nu P'_0(z) \right] \tilde{S}_1(\lambda_1 - \lambda_1 C(z)) \end{aligned} \quad (\text{A.11})$$

Equation (A.11) yields:

$$P'_0(z) = \frac{\lambda_0}{\nu} \left[ \frac{1 - [\tilde{S}_1(\lambda_1 - \lambda_1 C(z)) (\bar{p}\sigma \tilde{S}_2(\lambda_2 - \lambda_2 C(z)) (q \tilde{S}_3(\lambda_3 - \lambda_3 C(z)) + \bar{q}) + p\bar{\sigma} \tilde{S}_3(\lambda_3 - \lambda_3 C(z)) + (p\sigma + \bar{p}\bar{\sigma}))] \frac{C(z)}{z}}{[\tilde{S}_1(\lambda_1 - \lambda_1 C(z)) (\bar{p}\sigma \tilde{S}_2(\lambda_2 - \lambda_2 C(z)) (q \tilde{S}_3(\lambda_3 - \lambda_3 C(z)) + \bar{q}) + p\bar{\sigma} \tilde{S}_3(\lambda_3 - \lambda_3 C(z)) + (p\sigma + \bar{p}\bar{\sigma}))] - z} \right] P_0(z) \quad (\text{A.12})$$

which is the linear differential equation. On integrating (A.12), we obtain:

$$P_0(z) = P_0(1) \times \exp \left[ -\frac{\lambda_0}{\nu} \int_z^1 \frac{1 - [\tilde{S}_1(\lambda_1 - \lambda_1 C(u)) (\bar{p}\sigma \tilde{S}_2(\lambda_2 - \lambda_2 C(u)) (q \tilde{S}_3(\lambda_3 - \lambda_3 C(u)) + \bar{q}) + p\bar{\sigma} \tilde{S}_3(\lambda_3 - \lambda_3 C(u)) + (p\sigma + \bar{p}\bar{\sigma}))] \frac{C(u)}{u}}{[\tilde{S}_1(\lambda_1 - \lambda_1 C(u)) (\bar{p}\sigma \tilde{S}_2(\lambda_2 - \lambda_2 C(u)) (q \tilde{S}_3(\lambda_3 - \lambda_3 C(u)) + \bar{q}) + p\bar{\sigma} \tilde{S}_3(\lambda_3 - \lambda_3 C(u)) + (p\sigma + \bar{p}\bar{\sigma}))] - u} du \right] \quad (\text{A.13})$$

Using result from equation (A.8) into equation (A.5), we have:

$$(\theta - \lambda_1 + \lambda_1 C(z)) \tilde{P}_1(z, \theta) = (\tilde{S}_1(\lambda_1 - \lambda_1 C(z)) - \tilde{S}_1(\theta)) \left[ \frac{\lambda_0 C(z)}{z} P_0(z) + \nu P'_0(z) \right] \quad (\text{A.14})$$

Similarly, from equations (A.9)–(A.10) and equations (A.6)–(A.7) respectively, we get:

$$(\theta - \lambda_2 + \lambda_2 C(z)) \tilde{P}_2(z, \theta) = \bar{p}\sigma P_1(z, 0) (\tilde{S}_2(\lambda_2 - \lambda_2 C(z)) - \tilde{S}_2(\theta)) \quad (\text{A.15})$$

$$(\theta - \lambda_3 + \lambda_3 C(z)) \tilde{P}_3(z, \theta) = (qP_2(z, 0) + p\bar{\sigma}P_1(z, 0)) (\tilde{S}_3(\lambda_3 - \lambda_3 C(z)) - \tilde{S}_3(\theta)) \quad (\text{A.16})$$

Using the partial generating functions  $\tilde{P}_i(z, 0) = \sum_{n=0}^{\infty} z^n \tilde{P}_{i,n}(0)$ ,  $i = 1, 2, 3$  and equations (A.14)–(A.16), we finally obtain the equations (9)–(11) given in lemma 1.

*A.2 Proof of Theorem 1*

The probability generating function of the number of the customers in the orbit at an arbitrary epoch can be expressed as:

$$P(z) = P_0(z) + \tilde{P}_1(z, 0) + \tilde{P}_2(z, 0) + \tilde{P}_3(z, 0) \quad (\text{A.17})$$

Substituting the values of  $\tilde{P}_1(z, 0)$ ,  $\tilde{P}_2(z, 0)$  and  $\tilde{P}_3(z, 0)$  from equations (9)–(11) into equation (A.17) and doing some algebraic manipulations, we obtain the equation (12).

Using the normalising condition:

$$P(1) = P_0(1) + \tilde{P}_1(1, 0) + \tilde{P}_2(1, 0) + \tilde{P}_3(1, 0) \quad (\text{A.18})$$

and Lemma 1, we evaluate the value of  $P_0(1)$ .

*A.3 Proof of Lemma 2*

In order to obtain the values of  $P(I)$ ,  $P(E)$ ,  $P(S)$  and  $P(V)$ , in limiting case  $z \rightarrow 1$ , we apply L-hospital rule in equations (9)–(11) respectively and obtain the equations (13)–(16).

*A.4 Proof of Theorem 2*

The mean number of the customers in the orbit can be obtained by using:

$$E[N] = \sum_{n=0}^{\infty} nP_n = \lim_{z \rightarrow 1} P'(z)$$

L-hospital rule is used once to compute  $E[N]$ .

*A.5 Proof of Theorem 3*

The Little's formula is:

$$E[W] = \frac{E[N]}{\lambda E[X]} \quad (\text{A.19})$$

where

$$\lambda = \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3.$$

The mean waiting time in the orbit given in equation (18) can be easily obtained.